A REVIEW of PROBLEM-SOLVING CONCEPTS

and

A NEW LOOK AT LOGIC and PROOFS

for

SCIENCE or MATH STUDENTS

WHO HAVE PREVIOUSLY USED or MISUSED LOGIC

– NO-FRILLS VERSION –

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I. INTRODUCTION

Often students “mis-use” logic in their mathematics.

What do I mean? Sometimes they¹ incorrectly apply a logical concept; sometimes they apply a concept which they think is logical, but isn’t.

And on a different level, many times students misuse logical symbols.

These errors in thought lead to mistakes in both problem-solving and in the proof and understanding of theorems.

These are three areas which need our attention.

But before we can start talking about logic, we need some logic . . . .

FRAMEWORK for TALKING –

the “IF…, THEN…. ” STATEMENT

(CONDITIONAL STATEMENT or IMPLICATION).

In logic, when one says “If A, then C” what one means (asserts, avers, guarantees) is that “Whenever A occurs (takes place, is true, etc.), then it follows absolutely that B occurs (takes place, is true, etc.).” The sentence “If A, then C” says nothing at all about the truth or falsity of A; however, if (the sentence) is just designating or specifying a certain relationship between the truth values of A and C.

And in this pattern of thought “A” and “C” can stand for almost any two things, just so long as the “if…, then…..” sentence makes sense!

¹ I know, I know – I’m talking about you as if you weren’t here!
By the way, this pattern (which is called the **conditional statement**, “If A, then C.”) is one in which order is important. The “A” is called the **antecedent** and the “C” is called the **consequent**.

Thus, the two statements “If A, then C.” and “If C, then A.” have different meanings.

Example: Compare “If I go to my 9 am class, then I get out of bed by 8:45 am.” with “If I get out of bed by 8:45 am, then I go to my 9 am class.”

We’ll say more about the **conditional statement** and its **converse** later.

But now, what I really want to talk about first are some . . .

**ABSOLUTE BASIC CONCEPTS (ABCs) of LOGIC.**

I really think that the single most important logical idea when it comes to doing math is what they call the **Axiom of Replacement.** But before I can get to that, I need (logically) to start you off with the **Axiom of Identity.**

**The “Axiom of Identity.”**

In plain words, the **Axiom of Identity** says that a thing is what it is. To make this idea symbolic, if I use “X” and “Y” to stand for two things, and then if I say “X=Y” what I mean is that “X” and “Y” don’t stand for two separate things but represent two different names for the same thing. An important characteristic of this notion of **identity** is that it is **reflexive;** that is, it “goes both ways.” Thus I can symbolically state that

$$\text{If } X=Y, \text{ then } Y=X.$$  

For example: “If Robert is Bob, then it is equally true that Bob is Robert.”

“If $a = b + c$, then $b + c = a$.”

“If $\triangle ABC \cong \triangle DEF$, then $\triangle DEF \cong \triangle ABC$.”

(Recall that $\cong$ means “is congruent to.”)

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2 In many cases the “A” is called the **hypothesis** and the “C” is called the **conclusion.** Also, we say that “A is a sufficient condition for C” and that “C is a necessary condition for A.” (or “C is a necessary consequence of A.”)

3 “If A, then C.” is a **conditional statement**, and “If C, then A.” is called the **converse** of the conditional statement.

4 See footnote #3 above.

5 “Doing math” means not only working out solutions to problems, but also setting up problems and analyzing proposed final answers.

6 In some contexts this is called the “Axiom of Symmetry.”

7 The concept of congruent triangles is just a little bit different in that the two triangles really are different in that they are occupying different spaces, but the one is an exact replicate of the other in all side dimensions and in all angle measurements.
In math the reflexive nature of this axiom is often obscured by giving the “X=Y” and “Y=X” pairs completely different names and treating them as if they were totally different processes.

For example: We call \((a+b)^2 = a^2 + 2ab + b^2\) “expanding” or “FOIL-ing” the binomial, while we call \(a^2 + 2ab + b^2 = (a+b)^2\) “factoring the trinomial,” as if it were some new process. But isn’t it really just the “inverse of expanding,” or “un-expanding?”

And the recognition and appreciation of this fact and many other similar facts constitutes a step in the process of becoming “mathematically mature.” And as mathematical maturity increases, the student begins to see that math is not one jumble of disconnected factoids to be memorized for use in algebra class and another set of factoids to be used in trig class, etc., but that math is a process, implemented by really just a few general techniques and principles, by which human beings (a) investigate and pursue the consequences of mathematical thought (by creating new areas of math, stating theorems, and proving theorems) and (b) apply mathematical principles and techniques to the modeling and solution of physical problems.

**The “Axiom of Logical Equivalence.”**

A generalization of this “Axiom of Identity” is the “Axiom of Logical Equivalence.” This is a very general concept, but the way in which we’ll use it focuses on its mathematical aspect.

**Definition of Logical Equivalence for Equations:** Two equations are logically equivalent if and only if they have the same solution sets and if one equation can be derived from the other using the Basic Rules of Transformation for Equations, p. 6.

Note: In what follows, the math symbol “:=” is used to mean “... is defined to be ...” where the symbol to the left of := is a label or an abbreviation for the information to the right of :=.

For example: If we define \(E_1 := 2x + 3 = 5\) and \(E_2 := 2x = 2\), then in this example we can say that \(E_1\) and \(E_2\) are logically equivalent, (because \(E_1\) and \(E_2\) have the same solution set), and we write \(E_1 : E_2\). That is, in every-day terms, we say that \(2x + 3 = 5\) and \(2x = 2\) are logically equivalent.

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8 One of the goals of Precalculus is to begin the process of “mathematical maturation.”
9 One can analogously define Logical Equivalence for Pairs of Inequalities.
The “Axiom of Replacement.”

Now let’s talk about the “**Axiom of Replacement.**” Suppose that X and Y stand for equations. As we are using it here, this axiom states that if \( X : : Y \) (i.e., if \( X \) and \( Y \) are logically equivalent), then in any proof, solution, or verification where I see the statement (sentence, equation) \( X \), I can replace it with the statement (sentence, equation) \( Y \) without changing the outcome (solution, solution set, answer) of the problem.

Thus, if \( E_1 \) and \( E_2 \) are defined as in the example just above, then in any proof or solution where we have \( E_1 \) we may replace it with \( E_2 \) and not change the “outcome” or solution set. That is to say, any time you have \( 2x + 3 = 5 \) you can replace it with \( 2x = 2 \) and you know that you have not changed the solution set.

This is nothing new. You’ve been doing this for years. But now we are simply enunciating the process and trying to make it as clear as possible, in order to provide you with reasons why the process establishes a valid system for solving equations. You need to be able to have confidence in your work and to be able to justify your work results in the day-to-day critical peer review which occurs in the scientific / engineering workplace.

II. **THE REVIEW of SOLVING EQUATIONS**

We use the Axiom of Replacement repeatedly in the solution of an equation. And since solving equations is one of the skills that you need to have, we should probably systematically analyze the usual method of solution.

First of all, what does it mean to solve an equation? Well, let’s say that your universal set of interest is the set \( \mathbb{R} \) of real numbers. And for example, let’s say that your equation is \( 3x + 1 = 7 \). Then the solution (by which we really mean the solution set) to this equation is the subset of \( \mathbb{R} \) consisting of those numbers which makes the equation true upon substitution for \( x \). Thus the solution to this particular equation is

\[
\{ 2 \}\quad \text{(Notice that I box my final answer!)}
\]

How did I get this solution? By inspection! In effect, I just “asked” the equation, and it “told me.” But what if I ask it, and the answer doesn’t just jump out at me? Then I *simplify the equation by replacing it with a logically equivalent equation* which is, hopefully, more simple in that its solution set will jump right out at me!

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10 An equation is a mathematical sentence or statement; therefore, you will from time to time see these three words used synonymously: “equation,” “statement,” “sentence.”

11 By definition \( \mathbb{R} = \{ x \mid x \text{ is a real number} \} \). In interval notation this set is \( \left( -\infty, \infty \right) \).

12 This is the “McBee Method” of problem solving.
The usual progression of a solution might look something like this:

\[
3x + 1 = 7 \\
3x = 6 \quad (\text{subtract 1 from b.s.}) \quad 13 \\
x = 2 \quad (\text{divide b.s. by 2}) \\
\{2\}
\]

Here, of course, “b.s.” means “both sides.”

Notice that in each step above, the solution set for the resulting logically equivalent equation is a bit easier to see.

Now, let’s discuss the work above. Please understand that \( x = 2 \) is not the answer to anything – it is just another equation in the chain of simplifications using logically equivalent equations. But the equation \( x = 2 \) does have a solution set which actually jumps right into my lap!

The actual solution is the set of all numbers which make the original equation (and, consequently, all subsequent equations in the chain) true upon substitution. Namely, the solution is \( \{2\} \). How did I know that this is the solution? It is simple – I just asked \( x = 2 \), and it told me. And we can check our solution by substitution: \( 3(2) + 1 \) does equal 7.

You should carry out your chain of simplifications to the degree that your teacher requires; however, my own position on this matter is that you should write the solution set at the earliest instance that you are certain of the solution set. So if you “see” it at the \( 3x + 1 = 7 \) stage, go directly to \( \{2\} \).

Filling in all the steps is necessary at the MAT 1033 (Intermediate Algebra) level, but such detail may not be necessary at the MAC 2140 (Precalculus) level. However, if you don’t see the solution at the \( 3x + 1 = 7 \) stage, then by all means take it to the next stage and the next, if necessary, to make sure that you get the right solution set. Because, ultimately, we require correct answers.

When I am solving an equation, at each stage I pause for a moment and “ask” the equation “What is your solution?” If it “tells me,” that is, if the answer jumps out at me, then I write the solution set, mentally check it, and go on to the next problem. If it doesn’t “tell me,” then I go to the next stage of simplification and repeat the process.

Notice that in the paragraph above I said “pause for a moment” and that’s exactly what I mean – a moment. Not a minute. Not five minutes. Not a half-hour. Just pause for a moment. If the answer doesn’t jump right out at you, then go on to the next step of the

\[13 \text{ I keep telling you…. “b.s” means “both sides.”}\]
simplification process. However, if you need to do more steps, do them on paper; do not try to actually do steps in your head.

It’s no sin to write down lots of steps. I do it all the time. But if the answer jumps out at me I do not ignore it!

**RULES FOR PRODUCING EQUIVALENT EQUATIONS.**

Of course none of this is going to work if we replace one equation with another which is not logically equivalent.

Thus, it is in our best interest to have a clear set of rules for transforming one equation into a logically equivalent one. Here are the *Basic Rules of Transformation of Equations.*

**BASIC RULES of TRANSFORMATION — for EQUATIONS.**

- If you add the same number to both sides of an equation, then the resulting equation is logically equivalent to the first.

- If you subtract the same number to both sides of an equation, then the resulting equation is logically equivalent to the first.

- If you multiply both sides of an equation by the same non-zero number, then the resulting equation is logically equivalent to the first.

- If you divide both sides of an equation by the same non-zero number, then the resulting equation is logically equivalent to the first.

You know these rules; you’ve seen them many times before. But my advice to you here is that as you are simplifying an equation, you silently say the particular rule that you are using to yourself as you use it. This is your justification for the step you are taking. Doing it this way will mentally force you to pay attention to exactly what it is that you are doing. This will help reduce the number of careless errors that you make. \(^{14}\)

\(^{14}\) We all make careless errors. It is our job to reduce the number of careless errors that we make, and it is our goal to produce error-free work.
III. LOGIC

In mathematical logic we deal with things that are true, with things that are false, and with things that are conditional.

Here are three examples in the order listed above: $2 + 3 = 5$, $3 > 10$, $2x - 1 = 7$. We all know about “true” and “false.” But perhaps we need to say a little about “conditional.”

A *conditional sentence* is a (mathematical) sentence which contains a variable and which is true for some values of the variable and false for all other values of the variable.

In the following discussion our universal set will be $\mathbb{R}$, the set of *real numbers*, unless otherwise stated.

Let $C_1 := 3x + 5 > 11$. This symbolism is shorthand for: “$C_1$ is defined to be the open sentence $3x + 5 > 11$.” You have had to solve inequalities like this before. The instructions always were something like “Find the solution set in interval notation.” An alternate way of requesting this might be to say –

Find the **truth set** of $C_1$.

or

Find the **truth set** of $3x + 5 > 11$.

In any event, we would probably go through the following sequence of simplifications to arrive at the solution set (in interval notation) –

\[
\begin{align*}
3x + 5 &> 11 \\
3x + 5 - 5 &> 11 - 5 \\
3x &> 6 \\
\frac{3x}{3} &> \frac{6}{3} \\
x &> 2 \\
\end{align*}
\]

Thus, the solution set is $(2, \infty)$

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15 *A conditional sentence* is also often called an **open sentence**. I’ll use the two terms interchangeably.

16 Do you remember the *Rules for Producing Logically Equivalent Inequalities*?
Now, suppose that $C_1$ is defined as above and that $C_2$ is defined as follows:

$$C_2 := 5 - x > 2$$

In algebra class they might say “solve $3x + 5 > 11$ and $5 - x > 2$.” We have been taught that “and” means the “intersection” of two sets, so this problem could also be stated “Find $C_1 \cap C_2$.”

The point is that for a number $x$ to be in the solution set for $3x + 5 > 11$ and $5 - x > 2$, the number $x$ must be in BOTH the solution set for $C_1$ AND the solution set for $C_2$.

So to finish this problem we need to solve $5 - x > 2$:

\[
\begin{align*}
5 - x &> 2 \\
5 - x - 5 &> 2 - 5 \\
-x &> -3 \\
x &< 3
\end{align*}
\]

Thus, the solution set to $5 - x > 2$ is $(-\infty, 3)$.

And the final solution to $3x + 5 > 11$ and $5 - x > 2$ is $(-\infty, 3) \cap (2, \infty) = (2, 3)$

Thus, the final solution set to $3x + 5 > 11$ and $5 - x > 2$ is $(2, 3)$.

The problem just solved gives us an example of the logical concept “and.” And with this we can begin an elementary (and hopefully somewhat systematic) study of mathematical logic.

Logic has several ways of connecting ideas. Hopefully, we’ll be connecting true ideas, and that is our main goal. But in order to understand how true ideas can connect with one another to produce true results, we must also study false ideas and how they connect.

Well, let’s first get some technical definitions out of the way, so that we’ll all know what we are talking about.

**In mathematics**

A sentence is a declarative sentence. It has a subject and a verb and an object. It might be a sentence in the usual sense of the word; it might be an equation (the verb is “equals”); or it might be an inequality (the verbal phrase might be “is less than”).
An open sentence is a sentence containing a variable, such that for some values of the variable the sentence is true and for all other values of the variable the sentence is false.

A statement is a sentence that is either true or false, but not both at the same time.

Here are some examples:

1. The number 3 is a prime number.
2. \( \Delta ABC \) is isosceles.
3. \( x > 17 \).
4. For any natural number, \( n \), \( n^2 + n + 1 \) is a prime number.
5. \( 6x^2 - x - 1 = 0 \)
6. \( \frac{2x^2 - 5x + 3}{2x - 3} \)

#1, #2 and #4 are statements – #1 is true; #4 is false (although one might at first believe it to be true – try out some numbers, say \( n=1, 2, 3 \), and see what you get); and #2 is definitely either true or it is false, but I must know more about the triangle before I can determine the “truth value” of the statement.

#3 and #5 are open sentences, because in each case there is a “truth set” or “solution” to be found.

#6 is not even a sentence, it is simply an expression. Granted, it can be simplified, but it is still only an expression.

A simple statement is a statement which conveys only one idea.

A simple open sentence is an open sentence which conveys only one idea.

A compound statement is a statement made up of two or more statements joined by logical connectives.

A compound open sentence is an open sentence made up of two or more open sentences joined by logical connectives.
The logical connectives normally used in mathematical logic are

<table>
<thead>
<tr>
<th>Technical Name</th>
<th>Common Name</th>
<th>Symbol</th>
<th>Name of Symbol</th>
<th>Example(^\text{17})</th>
<th>Truth Criterion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conjunction ( Conj. )</td>
<td>And</td>
<td>(\land)</td>
<td>And</td>
<td>(p \land q)</td>
<td>Both conjuncts must be T.</td>
</tr>
<tr>
<td>Disjunction ( Disj. )</td>
<td>Or</td>
<td>(\lor)</td>
<td>Wedge</td>
<td>(p \lor q)</td>
<td>At least one of the disjuncts must be T.</td>
</tr>
<tr>
<td>Material Implication ( MI )</td>
<td>Implies</td>
<td>(\Rightarrow)</td>
<td>Arrow</td>
<td>(p \Rightarrow q)</td>
<td>The only false outcome is (T \Rightarrow F).</td>
</tr>
<tr>
<td>Material Equivalence ( ME )</td>
<td>…if and only if…. ( iff )</td>
<td>(\iff)</td>
<td>Double headed arrow</td>
<td>(p \iff q)</td>
<td>Both (p) and (q) match truth values.</td>
</tr>
<tr>
<td>Negation ( Neg. )</td>
<td>Not</td>
<td>(\sim)</td>
<td>Tilde</td>
<td>(\sim p)</td>
<td>Opposite truth value.</td>
</tr>
</tbody>
</table>

The first four connectives, Conj., Disj., MI, and ME, are called **binary operators**, because they operate on two symbols to produce a result. The last connective, Neg., is called a **unary operator**, because it operates on a single symbol.

**How the Logical Operators Determine the Truth Value of Compound Sentences**

The Truth Values for the basic logical connectives are shown in the following Truth Tables.

**Standard Basic Truth Tables for the Binary Operators – Conjunction, Disjunction, Material Implication, and Material Equivalence.**

Here’s how you set-up a basic truth table (for conjunction, disjunction, material implication, or material equivalence). Tables 1 & 2 show how to begin the set-up for the all four of the standard basic truth tables.

\(^{17}\) \(p\) and \(q\) stand for any two statements or open sentences.
Now we’ll give you the completed truth tables for each of the binary operators.

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>( p \land q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Table 3 – TT for Conjunction

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>( p \lor q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Table 4 – TT for Disjunction

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>( p \Rightarrow q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Table 5 – TT for Material Implication

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>( p \Leftrightarrow q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Table 6 – TT for Material Equivalence

The truth table for the unary operator, negation is given by

<table>
<thead>
<tr>
<th>p</th>
<th>(~ p)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Table 7 – TT for Negation

Here are some blank truth tables for you to practice on.
Truth Tables for More General Compound Sentences (Statements or Open Sentences).

First Type – Two Variables.

Suppose we are asked to analyze the truth values (via a truth table) of the compound statement \((p \land q) \Rightarrow q\). Each non-standard compound sentence such as this requires its own customized truth table. Here’s how you do it.

**How many rows?**
First count the number of distinct variables in the statement. In this example there are two, \(p\) and \(q\). You then raise 2 to this power, in this case \(2^2 = 4\). This gives you the total number of rows or “lines” or “scenarios” in your truth table.

**How many columns?**
You will have one column for each distinct variable and one column for each connective (logical operator). In this example, therefore, there will be 4 columns.
The “Final Column.”
The final column will be the complete compound sentence that you are analyzing. The heading of the final column will be the complete compound sentence, and the truth values (“Ts” or “Fs”) will show the truth or falsity of the sentence in each of the possible scenarios.

The “Preliminary Columns.”
The preliminary columns will systematically build up to the final column.

The Truth Values.
The truth values are “T” and “F.” The truth values in the “distinct variable columns” follow a rigid, predetermined pattern which defines the possible scenarios. The truth values in the “logical operator columns” are determined by our knowledge of the five basic truth tables defined above.

With these ideas in mind, let us now construct the –

Truth Table for \((p \land q) \Rightarrow q\)
I’ll demonstrate in steps. Of course, when you construct a truth table, you will just draw one table, and all we’ll see is the “final product.” (Table 11 in this display.)

<table>
<thead>
<tr>
<th>(p)</th>
<th>(q)</th>
<th>(p \land q)</th>
<th>((p \land q) \Rightarrow q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 9 – The Set-up for \((p \land q) \Rightarrow q\)

Explanation: Columns 1 & 2 are the “distinct variable columns.” Their truth values are always listed in the same way in order to define the four possible scenarios. Because of the parentheses I know that the MI (Material Implication) is the main connective – in other words, this sentence is an implication, and the Conj. (Conjunction) is a preliminary connective. This is why I have headed the logical operator columns as I have.

Now we fill out the preliminary column for \(p \land q\) according to our knowledge of the basic truth table for conjunction.
Finally, we fill out the final column for the implication \((p \land q) \Rightarrow q\) according to our knowledge of the basic truth table for material implication – recognizing that \(p \land q\) is the antecedent and \(q\) is the consequent in this case!

Table 11 tells us that the compound sentence \((p \land q) \Rightarrow q\) is true in every scenario; it is always true. Any such sentence that is always true – that can never be false – is called a tautology or a logically true sentence.

Thus, \((p \land q) \Rightarrow q\) is a tautology. However, it is not a very profound tautology! It simply says that

“If \(p\) and \(q\) are both true, then it follows that \(q\) is true.” \(^{18}\)

But it does make a good example of a tautology whose truth table is easy to write.

**Tautology, Self-Contradictory, and Contingent**

As we have just seen, a compound sentence can be a tautology (syn. logically true) if and only if all entries in the final column are “T.” On the other hand, suppose it turned out that all entries in the final column were “F.” In such a case we would call the sentence self-contradictory (syn. logically false). Finally, if the final column were a mix of “Ts” and “Fs,” then we’d say that the sentence were contingent.

\(^{18}\) Big deal! So what else is new?
An Interesting Principle
If you take two sentences and join them with an “iff” (which then becomes a ME sentence) and if this resulting ME turns out to be a tautology, then we say that the two sentences that you started out with are **logically equivalent**. They are essentially synonymous; they mean the same thing; one can be substituted for the other without changing the meaning.

Logical equivalence is good. And it is good to be able to check for logical equivalence. Let’s do some examples.

Problem: Are \( \sim (p \land q) \) and \( p \lor \sim q \) logically equivalent?
Solution: Construct the truth table for \( \sim (p \land q) \iff (\sim p \lor \sim q) \). If this ME turns out to be a tautology, then the two sentences are logically equivalent. If the ME is not a tautology, then the two sentences are not logically equivalent.

Since there are two distinct variables, we’ll again need four rows (scenarios) in our truth table. Also, we’ll need eight columns in our truth table (one each for the two distinct variables and six for the six logical connectives – three “tildes,” one “and,” one “wedge,” and one “double arrow.”)

It’s going to be a tight squeeze to get an 8-columned truth table across this page, with the final column heading \( \sim (p \land q) \iff (\sim p \lor \sim q) \).

Again, I’ll do several truth tables so that you can see the thought progression; whereas, in “real-time” you would just see the final product.

Can you see the logic in my column headings? In columns 3 & 4 I am building the left hand side of the ME; in columns 5, 6, & 7 I am building the right hand side of the ME; and in column 8 I am putting together the two sides with a “double arrow.”

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \land q )</th>
<th>( \sim (p \land q) )</th>
<th>( \sim p )</th>
<th>( \sim q )</th>
<th>( (\sim p \lor \sim q) )</th>
<th>( \sim (p \land q) \iff (\sim p \lor \sim q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 12**
Now I’m going to fill out column 3 using my knowledge of “and.”

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \land q$</th>
<th>$\neg (p \land q)$</th>
<th>$\neg p$</th>
<th>$\neg q$</th>
<th>$(\neg p \lor \neg q)$</th>
<th>ME</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td></td>
</tr>
</tbody>
</table>

Table 13

Next I fill out column 4 by doing a “not-column 3.” This gives me the truth values for the left hand side of the ME.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \land q$</th>
<th>$\neg (p \land q)$</th>
<th>$\neg p$</th>
<th>$\neg q$</th>
<th>$(\neg p \lor \neg q)$</th>
<th>ME</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td></td>
</tr>
</tbody>
</table>

Table 14

Next I start on building the right hand side of the ME. First thing I do is to complete column 5 by doing a “not-column 1.”

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \land q$</th>
<th>$\neg (p \land q)$</th>
<th>$\neg p$</th>
<th>$\neg q$</th>
<th>$(\neg p \lor \neg q)$</th>
<th>ME</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td></td>
</tr>
</tbody>
</table>

Table 15
Then I complete column 6 by doing a “not-column 2.”

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p ∧ q</th>
<th>¬(p ∧ q)</th>
<th>¬p</th>
<th>¬q</th>
<th>(¬p ∨ ¬q)</th>
<th>ME</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td></td>
</tr>
</tbody>
</table>

Table 16

And now I complete column 7 by doing an “or” to columns 5 & 6.

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p ∧ q</th>
<th>¬(p ∧ q)</th>
<th>¬p</th>
<th>¬q</th>
<th>(¬p ∨ ¬q)</th>
<th>ME</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
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<tr>
<td>T</td>
<td>F</td>
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<td>T</td>
<td></td>
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<tr>
<td>F</td>
<td>T</td>
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<td></td>
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<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td></td>
</tr>
</tbody>
</table>

Table 17

Finally, to complete column 8 (and in fact the entire truth table) I do an “iff” to columns 4 & 7, which means that compare the truth values in each row (scenario) to see if they match. I’ve highlighted cols. 4 & 7 to focus my attention – On paper I would normally circle these columns. At any rate, as you can see, the truth values match one-on-one. Thus the final column has all “Ts.” Thus the ME, ¬(p ∧ q) ⇔ (¬p ∨ ¬q) is a tautology.

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p ∧ q</th>
<th>¬(p ∧ q)</th>
<th>¬p</th>
<th>¬q</th>
<th>(¬p ∨ ¬q)</th>
<th>ME</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
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<tr>
<td>T</td>
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<tr>
<td>F</td>
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<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Table 18
Thus, we have proven that $\sim (p \land q)$ and $\sim p \lor \sim q$ are logically equivalent. This logical equivalence is a part of a rule of logic known as **DeMorgan’s Law**, which states that

$\sim (p \land q)$ is logically equivalent to $\sim p \lor \sim q$

and

$\sim (p \lor q)$ is logically equivalent to $\sim p \land \sim q$.

DeMorgan’s Law is very important. The ideas involved come up quite frequently in higher-level mathematical thinking. In the first form it says in plain words that if the compound sentence “p and q” is false, then either p is false or q is false and vice-versa – if either p is false or q is false or both, then “p and q” is false. In the second form it makes an analogous statement about “p or q.”

**Next Example**

In my next example I want to show you why a conditional statement and its converse are not logically equivalent.

Thus we examine $p \Rightarrow q$ and $q \Rightarrow p$. To determine their logical relationship we form the ME $(p \Rightarrow q) \leftrightarrow (q \Rightarrow p)$ and analyze it via a truth table.

Again there are only two distinct variables, so our truth table will have 4 rows. There are three connectives, so our truth table will have $3^2 = 9$ columns.

Here follows my analysis, again in expanded form.

Again, and let me stress this point, the arrangement of “Ts” and “Fs” in the first two columns is standard. This is the required order for your set-up.

Columns 3 & 4 build the left and right sides of the ME, which is the final column.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \Rightarrow q$</th>
<th>$q \Rightarrow p$</th>
<th>$(p \Rightarrow q) \leftrightarrow (q \Rightarrow p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Table 19
Now I complete the third col. according to the MI rule with $p$ the antecedent and $q$ the consequent.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \Rightarrow q$</th>
<th>$q \Rightarrow p$</th>
<th>$(p \Rightarrow q) \Leftrightarrow (q \Rightarrow p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 20

Next I complete the fourth column according to the MI rule, but this time with $q$ the antecedent and $p$ the consequent. If you are at all dyslexic, you must be very careful right here! I know. I am. 19

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \Rightarrow q$</th>
<th>$q \Rightarrow p$</th>
<th>$(p \Rightarrow q) \Leftrightarrow (q \Rightarrow p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Table 21

Now is the easy part – I just compare columns 3 & 4. In rows that match the ME is “T.” In rows that don’t match, the ME is “F.” It’s just that simple.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \Rightarrow q$</th>
<th>$q \Rightarrow p$</th>
<th>$(p \Rightarrow q) \Leftrightarrow (q \Rightarrow p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

19 … Both dyslexic and careful!
As you can see, the final column is not all “Ts.” Thus the ME is not a tautology.\textsuperscript{20} Thus the two sentences $p \Rightarrow q$ and $q \Rightarrow p$ are not logically equivalent. Thus they do not mean the same thing. Thus they cannot be used interchangeably!

So never confuse a conditional statement with its converse.

**Here are some Important Logically Equivalent Statements**

(Remember that $A : : B$ means that $A$ and $B$ are logically equivalent, \textit{i.e.} the sentence $A \Leftrightarrow B$ is a tautology.)

1. $\sim (p \land q) : : (\sim p \lor \sim q)$
2. $\sim (p \lor q) : : (\sim p \land \sim q)$
3. $(p \Leftrightarrow q) : : ((p \Rightarrow q) \land (q \Rightarrow p))$
4. $(p \Rightarrow q) : : (\sim p \lor q)$
5. $\sim (p \Rightarrow q) : : (p \land \sim q)$

**Discussion of These Logical Equivalences**

- The first and second are, of course, DeMorgan’s Laws.
- The third is that an “iff” statement means that both an implication and its converse are true.
- The fourth, stated in words, is “p implies q means the same thing as saying that either p is false or q is true (or both).\textsuperscript{21}
- The fifth shows that the negation of a conditional (that is – a conditional is false) is when the antecedent $p$ is true and the consequent $q$ is false. But, thinking back to the basic truth table for MI, we already knew this.

\textsuperscript{20} This ME is not a tautology. But also observe that it is not self-contradictory. It is in fact a contingent statement.

\textsuperscript{21} The example that I’ve given for years shows the obvious equivalence of the following two statements…. (You must imagine this is from a scene in a Cowboy movie). The good guy is holding a pistol on the bad guy. Does the good guy say “If you move, then I’ll shoot you.” or does he say “Don’t move or I’ll shoot you.”? I’m not advocating violence, mind you, but this example does show the logical equivalence of the two statements.
A Second Type of Compound Sentence
I don’t want to carry this too far, but I do think that it will be instructive if I show you the analysis of a compound sentence having more than two distinct variables. Let’s do one with three distinct variables.

These longer sentences are no more difficult to do; they just take longer. Every time I increase the number of distinct variables by one, I double the number of rows in the truth table, because every variable has two truth values, “T” and “F.”

The example that I want to do is \((p \land q) \Rightarrow r\).

Here is the set-up. Notice that in the eight rows for the distinct variables there is a definite pattern. Also note that there are three distinct variables and two connectives; thus, there are five columns in this truth table.

<table>
<thead>
<tr>
<th>(p)</th>
<th>(q)</th>
<th>(r)</th>
<th>(p \land q)</th>
<th>((p \land q) \Rightarrow r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td></td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td></td>
<td></td>
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<td>T</td>
<td>F</td>
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<td>F</td>
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<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 23

To determine the truth values in the fourth column, I simply focus on the \(p\) and \(q\) columns and recall the truth table for Conj. (“and”).

<table>
<thead>
<tr>
<th>(p)</th>
<th>(q)</th>
<th>(r)</th>
<th>(p \land q)</th>
<th>((p \land q) \Rightarrow r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td></td>
<td></td>
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<tr>
<td>T</td>
<td>F</td>
<td>T</td>
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<td></td>
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<tr>
<td>T</td>
<td>F</td>
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<td>F</td>
<td>T</td>
<td>T</td>
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<td>T</td>
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<td>F</td>
<td>F</td>
<td>T</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 24

This last column was for me the “hard one,” because the antecedent is in the 4\(^{th}\) col. and the consequent is in the 3\(^{rd}\) col. Thus, I had to “think backwards” to evaluate the “arrow.”
From Table 25 you can see that the statement \((p \land q) \Rightarrow r\) is contingent.

However it is interesting to note that the statement \(p \Rightarrow (q \Rightarrow r)\), which is also contingent, has the same truth table. That is to say, it has the same final column in its truth table.

Now what this means is that if I take the ME of these two statements, I will find that the truth table of the ME is a tautology. Thus, it can be shown that the two statements are logically equivalent, \(i.e.\) \(\left((p \land q) \Rightarrow r\right) : \left(p \Rightarrow (q \Rightarrow r)\right)\).

Thus, rather than building one massive truth table for the ME, I can build two smaller truth tables, one for each “half” and then compare the final columns of the two truth tables. The two statements are logically equivalent if and only if the two final columns match identically. \(^{22}\)

So with this in mind, I challenge you to write the truth table for \(p \Rightarrow (q \Rightarrow r)\) and compare it with the truth table that I wrote for \((p \land q) \Rightarrow r\). I’m even supplying you with a partially constructed truth table just so we’ll have all eight scenarios the same. You do the rest. Good Luck!

\(^{22}\) Of course this “if and only if” presumes that you don’t make any mistakes in constructing the truth tables!
IV. PROOFS

One skill expected of the critical-thinking, mathematically-mature Calculus student is the ability to prove elementary theorems. This skill is logic-based in as much as a proof is a logical argument. Logical arguments have various forms many of which must be known by the student in order to successfully and efficiently prove theorems. Here are some patterns of reasoning often used in proving theorems.

1. **Modus ponens**

\[
p \Rightarrow q
\]

\[
p
\]

\[
q
\]

Here’s how you read this: If “p implies q” is true and “p” is true, then it follows that “q” is true.

“p implies q” is the **first premise**, “p” is the **second premise**. And “q” is the **conclusion**.

2. **Modus tollens**

\[
p \Rightarrow q
\]

\[
\sim q
\]

\[
\sim p
\]

Here’s how you read this: If “p implies q” is true and “q” is false, then it follows that “p” is false.
“p implies q.” is the first premise. “not-q” is the second premise. And “not-p” is the conclusion.

3. **Pure Hypothetical Syllogism**

\[
\begin{align*}
p & \Rightarrow q \\
q & \Rightarrow r \\
p & \Rightarrow r
\end{align*}
\]

Here’s how you read this: If ‘p implies q” is true and if “q implies r” is true, then it follows that “p implies r” is true.

“p implies q.” is the first premise. “q implies r” is the second premise. And “p implies r” is the conclusion.

4. **Disjunctive Syllogism**

\[
\begin{align*}
p \lor q \\
\sim p \\
q
\end{align*}
\]

Here’s how you read this: If “p or q” is true and “p” is false, then it follows that “q” is true.

“p or q.” is the first premise. “not-p” is the second premise. And “q” is the conclusion.

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**Proof Based Upon Definition.**

Your complete understanding of definitions many times aids in proofs. For example, let’s prove the little theorem

Theorem: The sum of two odd natural numbers is even.

Proof:

(1). The set of natural numbers is \( \mathbb{N} = \{1, 2, 3, 4, \ldots\} \). (Def: Natural Numbers)

(2). Let \( m \in \mathbb{N} \) be an odd number, then for some \( k \in \mathbb{N}, \ m = 2k - 1 \) (Def: Odd number)

(3). Let \( n \in \mathbb{N} \) be an odd number, then for some \( l \in \mathbb{N}, \ n = 2l - 1 \) (Def: Odd number)

(4). Consider \( m + n \) (Let’s add the numbers.)

\[ m + n = (2k - 1) + (2l - 1) \] (Law of Subs.)
\[= 2k + 2l - 2 \quad \text{(Simple algebra)}\]
\[= 2(k + l - 1)\]

(5). \(k + l - 1 \in \mathbb{N}\)  
\((k \text{ is a whole number greater than or equal to } 1.\)  
\(l \text{ is a whole number greater than or equal to } 1.\)  
\(k+l \text{ is a whole number greater than or equal to } 2.\)  
Therefore \(k+l-1\) is a whole number greater than or equal to 1.)

(6). Therefore \(2(k + l - 1)\) is an even number  \(\text{(Def: Even number)}\)

(7). Therefore \(m + n\) is an even number. \(\text{(Transitive Property of Equality)}\) \(^{23}\)

\(^{23}\) The Transitive Property of Equality states that if \(A=B\) and \(B=C\), then \(A=C\). We use this all the time without ever actually giving it credit.