ANOTHER LIMIT EXAMPLE

Prove the limit: \( \lim_{x \to -2} (x^2 - x + 1) = 7 \).

**Proof:** [1] Let \( \varepsilon > 0 \) be assigned. We must find \( \delta > 0 \) such that

\[
0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon .
\]

In our problem

\[
0 < |x + 2| < \delta \Rightarrow |(x^2 - x + 1) - 7| < \varepsilon .
\]

(1.1)

[2] **Work:** Consider

\[
|(x^2 - x + 1) - 7| = |x^2 - x - 6| = |x - 3| \cdot |x + 2|
\]

(1.2)

Observe that we can get a “handle” (a “\( \delta \)-handle” at that!) on \( |x + 2| \), but we have no “handle” on \( |x - 3| \). We need to create a “handle” for \( |x - 3| \).

To this end, we require that

\[
|x + 2| < 1
\]

(1.3)

How is it that we can make this requirement? Well, we are given that \( x \) is approaching negative 2. So as it does, sooner-or-later it must come within one unit of negative 2. That’s what “inequality (1.3)” says. Thus the requirement of “inequality (1.3)” is merely saying that we are going to restrict our investigation of this limit process to its end-game, its “final quarter,” the “NBA play-offs,” so to speak. It is as if there were this giant asteroid (that would be \( x \)), crashing down toward earth (that would be negative 2), and we scientists were interested in it only after it hit the atmosphere (a distance 1 from negative 2)! Get the picture? I hope so.

Next issue – how do we use (1.3) to get a handle on \( |x - 3| \)? Here is your answer:

1. We expand the absolute value inequality.

\[
|x + 2| < 1 \quad \Leftrightarrow \quad -1 < x + 2 < 1
\]

(1.4)

2. We **subtract** 5 from all three quantities

\[
-1 - 5 < x + 2 - 5 < 1 - 5
\]

\[
\therefore \quad -6 < x - 3 < -4
\]

(1.5)

to give us x-3 “in the middle.”
3. But remember, we wish to capture the absolute value of \( x-3 \), and there is a neat “trick” that enables us to do this. Multiply all three entries of (1.5) by negative 1, producing

\[
6 > 3 - x > 4
\]

which can be re-written

\[
4 < 3 - x < 6.
\]  

(1.6) \hspace{1cm} (1.7)

Now the LHS (left hand side) of (1.7) tells us that \( 3 - x \) is positive (since it is greater than 4). Therefore,

\[
3 - x = |3 - x|
\]  

(1.8)

but \(^1\)

\[
|3 - x| = |x - 3|
\]  

(1.9)

so we can re-write (1.7) as

\[
4 < |x - 3| < 6
\]  

(1.10)

and the RHS (right hand side) of (1.10) will provide us with the desired “handle” on \( |x - 3| \). That is,

\[
|x - 3| < 6
\]  

(1.11)

Now we can complete our chain of reasoning begun in (1.2). We have

\[
|x + 2| < \delta
\]  

(1.12)

and, combining this with (1.11), we can “finish” (1.2)

\[
|(x^2 - x + 1) - 7| = |x^2 - x - 6| = |x - 3| \cdot |x + 2| < 6\delta.
\]  

(1.13)

Thus it appears that if we choose \( \delta \) so that both \( \delta \leq 1 \) and \( 6\delta = \varepsilon \) we shall be able to verify the limit. To this end, we

\[
\text{let } \delta = \min \left\{ 1, \frac{\varepsilon}{6} \right\}
\]

\(^1\) Remember: \( |A - B| = |B - A| \)

\(^2\) Please see the “NOTE” at the end of this paper for a “geometric” explanation of \( |x - 3| < 6 \).
**[3] Proof (Verification):** Let $\varepsilon > 0$ be assigned, and let $\delta = \min \left\{ 1, \frac{\varepsilon}{6} \right\}$. Then if $0 < |x+2| < \delta$, we have

(a) $|x+2| < 1$, so it follows$^3$ that $|x-3| < 6$, and

(b)

\[
\left| (x^2 - x + 1) - 7 \right| = |x^2 - x - 6| = |x - 3| \cdot |x + 2| < 6\delta \leq 6 \cdot \frac{\varepsilon}{6} = \varepsilon \quad (1.14)
\]

Thus

\[
\left| (x^2 - x + 1) - 7 \right| < \varepsilon \quad (1.15)
\]

Hence, by the $\varepsilon$, $\delta$ – definition,

\[
\lim_{x \to 2} (x^2 - x + 1) = 7 \quad (1.16)
\]

Note: if $0 < |x+2| < 1$, then $x \in (-3, -2) \cup (-2, -1)$, and as you can see from the graph, the *distance* between $x$ and 3 must be less than 6. Thus $|x-3| < 6$.

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$^3$ See the Work area for details.