Part #1: Something Old and Something New.

#1A Example: Prove \( \lim_{x \to 3} x^3 = 27 \).

Proof: [1]. We must show that for every \( \varepsilon > 0 \), there exists a corresponding \( \delta > 0 \) such that \( 0 < |x - 3| < \delta \) implies that \( |x^3 - 27| < \varepsilon \). This concept, pronounced as in the sentence above, looks like this in the following in mathematical/logical symbolism –

\[
\forall \varepsilon > 0, \exists \delta > 0 : 0 < |x - 3| < \delta \Rightarrow |x^3 - 27| < \varepsilon .
\]

(See the footnote below for definitions.1)

[2]. Work: Consider the LHS of the epsilon inequality:

\[
|x^3 - 27| = |x - 3||x^2 + 3x + 9| .
\]

\[
[[ \text{ Recall that } A^3 - B^3 = (A - B)(A^2 + AB + B^2) ]] \]

And we can get a “handle” on the \( |x - 3| \) with our delta inequality. But it remains to be seen just how we can get a “handle” on the \( |x^2 + 3x + 9| \). Here’s one way2 to do it –

If \( x \) is getting “closer-and-closer” to 3, then sooner or later it’s got to get closer to 3 than 1 unit. So we can take a “preliminary” delta, call it \( \delta_1 \), which is equal to 1. i.e., Let \( \delta_1 = 1 \), and require that \( 0 < |x - 3| < \delta_1 = 1 \). So what are the ramifications of this requirement?

\[
|x - 3| < 1 \Rightarrow -1 < x - 3 < 1 \Rightarrow -1 + 3 < (x - 3) + 3 < 1 + 3 \Rightarrow 2 < x < 4 . \quad \text{ (See footnote 3)}
\]

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1 “\( \forall \)” means “for each,” “for every,” or “for all,” depending upon the context.

“\( \exists \)” means “there is,” or “there exists,” depending upon the context.

"\( \exists \)" or "\( \forall \)" means “such that,” or “with the property that.”

"\( \Rightarrow \)” means “implies,” “leads to,” or "... then,” as in “if . . . , then . . . .”

Please note that "\( = \)" and "\( \Rightarrow \)" are NOT, I repeat, NOT synonyms. They do not mean the same thing. They cannot be used interchangeably, and they will not be used interchangeably. Am I clear on this issue?

2 I say “one way” because, especially in analysis, there’s usually more than one (correct) way to proceed.

3 Notice that, technically, \( 0 < |x - 3| < \delta_1 = 1 \) means, in interval notation, that \( x \in (2, 3) \cup (3, 4) \).

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From this chain of reasoning we draw two further conclusions:

\[
\begin{align*}
(a) \quad |x^2| &= x^2 < 16 \quad \text{and} \\
(b) \quad |3x| &= 3x < 12.
\end{align*}
\]

(See footnote \(^4\)).

\[2 – a\] An “ASIDE.” Next we need to recall the “triangle inequality,” which you may or may not have studied in college algebra, precalculus, and/or in trig\(^5\). In **words**, the triangle inequality says that

“the shortest distance between two points is along a straight line.”

In symbols it says

\[
|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|
\]

where . . . . \(^6\)

Graphically, we “see” the triangle inequality as

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\(^4\) From \(2 < x < 4\) we can conclude that \(x\) is positive and therefore that \(x = |x|\) and that \(x^2 < 16\). Consequently, we deduce that \(|x^2| < 16\). In a similar manner we deduce that \((|x|)(x < 4) \Rightarrow (|3x| < 12)\). [[Remember: “\(\wedge\)" means “and” AND “\(\Rightarrow\)” means “implies.”]]

\(^5\) I feel certain that you studied the triangle inequality at least in trig, because it’s most easily understood in the context of vectors, and vectors were studied in trig. However, you may not have appreciated the significance or importance of the triangle inequality at that time; it may have been just something else to memorize.

\(^6\) \(\vec{a}\) and \(\vec{b}\) are vectors, \(|\vec{a}|\) and \(|\vec{b}|\) are their lengths (or “magnitudes,” as we call them), \(\vec{a} + \vec{b}\) is the “vector sum” of the two vectors, and, finally, \(|\vec{a} + \vec{b}|\) is the length (magnitude) of the vector sum.
where the length of the vector $\overrightarrow{a} + \overrightarrow{b}$ (the vector is indicated by bold $\mathbf{a+b}$ in the graph) is obviously less than the sum of the lengths of the other two vectors$^7$.

Now, if our symbols represent numbers on the number line (instead of vectors in 2-space), then the triangle inequality looks like this:

$$|a + b| \leq |a| + |b|.$$ 

The triangle inequality is very important in mathematical analysis. So I think that this little “excursion,” or “aside,” is time well-spent. Besides, we’re going to use it just now in our proof$^8$.

... [end of the “ASIDE.”]

So let’s do a quick review of where we are . . . .

We’ve got a “delta handle” on $|x - 3|$ and we’re trying to get some sort of “handle” on $|x^2 + 3x + 9|$. Here’s what we’re going to do – we’ll use the triangle inequality twice and then we’ll use conclusions (a) and (b) – see the top of page 2. . . . Here goes!!!

$$|x^2 + 3x + 9| \leq |x^2| + |3x| + |9| < 16 + 12 + 9 = 37. \quad (***)$$

So now if we put together (*) and (**), we get

$$|x^3 - 27| = |x - 3||x^2 + 3x + 9| < 37\delta. \quad (***)$$

Now since we want to find a delta such that $|x^3 - 27| < \varepsilon$ it just seems to make good sense to put $37\delta = \varepsilon$ and solve for delta, getting $\delta = \frac{\varepsilon}{37}$. But we can’t forget that to even get this far we had to already assume ( “demand,” “require” ) that delta had to be 1 or smaller. Thus to insure that (***) can really happen, we need our “final delta” to be the smaller of the two numbers 1 and $\frac{\varepsilon}{37}$. And the way we do this mathematically is to say “let $\delta = \min \left\{ 1, \frac{\varepsilon}{37} \right\}$.”

... [end of the “WORK.”]

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$^7$ Now I hope that I’m not insulting your intelligence by remarking that my “picture” hasn’t proven anything at all. But it does give a fairly powerful feeling that the triangle inequality is “true,” and hence “provable.”

$^8$ If you’re good, then maybe one day I’ll prove the triangle inequality for you!
[3]. Proof: Let \( \varepsilon > 0 \) be assigned. Then we let \( \delta = \min \left\{ 1, \frac{\varepsilon}{37} \right\} \), and now we establish our required implication:

IF \( 0 < |x - 3| < \delta \),

THEN (1) \( |x - 3| < 1 \) and it follows that

(a) \( |x^2| = x^2 < 16 \) and (b) \( |3x| = 3x < 12 \).

AND (2) \( |x^3 - 27| = |x - 3||x^2 + 3x + 9| \leq \delta |x^2 + 3x + 9| \leq \delta \left(|x^2| + |3x| + 9\right) < \delta (16 + 12 + 9) < 37\delta \leq 37 \frac{\varepsilon}{37} = \varepsilon.

[4]. In Summary: We have shown that . . .

“for any epsilon greater than zero, there exists a corresponding delta greater than zero such that whenever \( x \) is sufficiently close to 3 – but not equal to 3 (namely when \( 0 < |x - 3| < \delta \)) it follows (. . . as the night the day, . . . ) that \( x^3 \) will be closer to 27 than any prescribed epsilondistance (\( |x^3 - 27| < \varepsilon \))."

Thus,

\[
\lim_{x \to 3} x^3 = 27
\]

#1B Problems: (1B.1) Prove \( \lim_{x \to 2} x^3 = 8 \)

(1B.2) Prove \( \lim_{x \to 2} x^3 + 2x = 12 \) (Hint 10).

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9 Polonius to his son Laertes . . .
This above all, – to thine own self be true,
And it must follow, as the night the day,
Thou canst not then be false to any man.
Farewell: my blessing season this in thee!

-- William Shakespeare (Hamlet, Act I, Scene III)

10 Hint – Factor by grouping as follows:
\[
|x^3 + 2x - 12| = |(x^3 - 8) + (2x - 4)| = |x - 2| |x^2 + 2x + 4| = |x - 2| |x^2 + 2x + 6|
\]
Part #2: Something Borrowed.

#2A—Development of the Ideas:

As you have learned in MAT 1033 & MAC 1105, and as we have already gone-over time after time in this class, the **slope of a (straight) line is constant**. That is to say, it is the same at any point along the line. And it is the “rise over the run,” the “change in y over the change in x.” In symbols,  
\[ m = \frac{y_2 - y_1}{x_2 - x_1}, \]

or in a slightly more compact notation,  
\[ m = \frac{\Delta y}{\Delta x}. \]

Here \( m \) is the slope of the line at point \( P(x_1, y_1) \) and \( Q(x_2, y_2) \) is any other point on the line.

Things get a little more complicated when the graph is not a straight line, but we still want to know the slope of a line tangent to the graph at a point \( P(x_1, y_1) \) on the graph \( \Gamma \).\(^{11}\) Actually, what we’ll be looking for, much of the time, is the equation (in standard form) for the line tangent to the graph of \( y = f(x) \) at the point \( P(x_1, y_1) \in \Gamma \).

Now, the slope of a secant line between two points \( P(x_1, y_1) \in \Gamma \) and \( Q(x_2, y_2) \in \Gamma \) is given by  
\[ m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h} = \frac{f(a + h) - f(a)}{h} \]

where  
\[ y_2 = f(x_2), \]
\[ y_1 = f(x_1), \]
\[ x_2 = x_1 + h, \text{ and} \]
\[ a = x_1. \]

So, the slope of the line tangent to \( \Gamma \) at \( P(a, f(a)) \) is given by  
\[ m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}, \text{ if this limit exists.} \]

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\(^{11}\) I think that I’m just going to go ahead and use some “advanced” notation with you folks. If \( y = f(x) \) is a function, then the graph of the function will be denoted by \( \Gamma_f \) or simply by \( \Gamma \) if there can be no confusion about which graph we’re referring to. – And please note that \( \Gamma \) (gamma) is the letter G in the Greek alphabet, and G is the first letter of the word “graph.”
OK, so now we know how to find the slope of the tangent line. Next we need to find the equation of the tangent line. Well, we were given a point on the tangent line, namely \( P(a, f(a)) \), and we have just calculated the slope \( m \); thus, we should be able to substitute these quantities into the familiar "point-slope" equation,

\[
y - y_1 = m(x - x_1),
\]

getting

\[
y - f(a) = m(x - a). 
\]

Finally, we want to put this linear equation into standard form, which for this class means

\[
Ax + By + C = 0 \quad \text{or} \quad Ax + By = D
\]

with the stipulations that

- the “x”-term comes first,\(^{12}\)
- the “leading coefficient,” \( A \), must be a non-negative integer\(^{13}\), and
- the equation must be “cleared of fractions.”\(^{14}\)

These are the rules of the game for this activity.

So, let’s see how they “work.”

**Example Problem:** Find the equation in Standard Form for the line tangent to the graph of the given equation at the given number \( a \).

\[
y = x^2 - 4x + 5 \quad a = 3.
\]

**Sol:** [1]. Find \( m \) (using 5-step method.)

1. \( f(a) = f(3) = (3)^2 - 4(3) + 5 = 9 - 12 + 5 = 2 \)
2. \( f(a + h) = f(3 + h) = (3 + h)^2 - 4(3 + h) + 5 = 9 + 6h + h^2 - 12 - 4h + 5 = 2 + 2h + h^2 \)
3. \( f(a + h) - f(a) = f(3 + h) - f(3) = (2 + 2h + h^2) - 2 = 2h + h^2 \)
4. \( \frac{f(a + h) - f(a)}{h} = \frac{2h + h^2}{h} = 2 + h \)

\(^{12}\) Example: \( 3y - x + 2 = 0 \) is NOT in Std. Form.

\(^{13}\) \( A \in \mathbb{Z^+} \cup \{0\} = \{0, 1, 2, 3, 4, 5, \ldots\} \)

\(^{14}\) Example: \( 3x - \frac{\sqrt{2}}{5}y = \frac{1}{2} \) must be “cleared” to \( 30x - 2\sqrt{2}y = 5 \) or \( 30x - 2\sqrt{2}y - 5 = 0 \).
(5). \[ m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} (2 + h) = [2] \]

[2]. Substitute \( m \) and \( a \) into the P-S (Point-Slope) Form:

\[ y - f(a) = m(x - a) \]

getting

\[ y - 2 = 2(x - 3) \quad (*) \]

And

[3]. Translate (*) into Std. Form.

\[ y - 2 = 2(x - 3) \Rightarrow y - 2 = 2x - 6 \Rightarrow 0 = 2x - y - 6 + 2 = 2x - y - 4, \]

\[ \therefore 2x - y - 4 = 0 \]

is the desired equation.

#2B. Now do these problems.\(^{15}\) Find the equation of the line tangent to the graph of the given equation at the given number \( a \).

(2B-1). \[ y = x^2 - 2x + 4 \quad a = 4 \]

(2B-2). \[ y = -x^2 - 4x \quad a = -4 \]

(2B-3). \[ y = \sqrt{x-1} \quad a = 2 \]

(2B-4). \[ y = \frac{1}{x^2} \quad a = 2 \]

(2B-5). \[ y = \frac{1}{\sqrt{x}} \quad a = 9 \]

End of “black dog calculus part02”

\(^{15}\) Folks, this IS the Black Dog Section!