THINGS THAT YOU SHOULD KNOW.

I. PASCAL’S TRIANGLE

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Can you see how to construct the next row of the “Triangle?” Would it be 1, 6, 15, 20, 15, 6, 1?

“Pascal’s Triangle” gives the numerical coefficients for the expansion of a binomial such as $(x + y)^n$. For example

$$(x + y)^4 = 1x^4y^0 + 4x^3y^1 + 6x^2y^2 + 4x^1y^3 + 1x^0y^4$$

uses the coefficients from the 5-th row of the “Triangle.”

It is a time-saving skill to be able to quickly and efficiently expand binomials to various powers during the construction of many of the more common and important difference quotients with which we’ll be dealing.

II The “Method” of Rene Descartes.

In his “Method,” Descartes enumerates four steps that he takes in his investigations:

The first was never to accept anything for true which I did not clearly know to be such; that is to say, carefully to avoid precipitancy and prejudice, and to comprise nothing more in my judgement than what was presented to my mind so clearly and distinctly as to exclude all ground of doubt.

The second, to divide each of the difficulties under examination into as many parts as possible, and as might be necessary for its adequate solution.

The third, to conduct my thoughts in such order that, by commencing with objects the simplest and easiest to know, I might ascend by little and little, and, as it were, step by step, to the knowledge of the more complex; assigning in thought a certain order even to those objects which in their own nature do not stand in a relation of antecedence and sequence.
And the last, in every case to make enumerations so complete, and reviews so general, that I might be assured that nothing was omitted.¹

I believe that this “Method” can help us in our mathematical investigations, some 369 years later.

II. Limits of Difference Quotients.

We'll need lots of practice at calculating and simplifying Difference Quotients and then “taking” their limits.

Following Descartes lead, I would like us to break the process down into five steps. Thus, the method that we will use is called the **Five-Step Method**.

Here’s the “Plan:“

Set up the DQ, algebraically reduce it, and then state the limit using the “Big 11” Laws. Recall that

$$DQ = \frac{f(a+h) - f(a)}{h} \quad \text{and} \quad m_p = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

where $P(a, f(a))$ is on the graph of $f(x)$ and $m_p$ is the slope of the line tangent to the graph of $f(x)$ at the point $P(a, f(a))$.

Let me show you some examples.

1. **Given**: $f(x) = x^2 + 3x$ and $a = 5$.
   **Required**: Find $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ using the 5-Step Method, and interpret this result.

   **Sol.**²

   [1]. $f(a) = f(5) = 25 + 15 = 40$

   [2]. $f(a+h) = f(5+h) = (5+h)^2 + 3(5+h) = 25 + 10h + h^2 + 15 + 3h = 40 + 13h + h^2$


² Note: I'm not circling my results for each step of the 5-Step Method in this paper, because it's a pain in the neck to do so in “Word.” But I really want you to circle the results of the first 4 steps on your quizzes and tests, because it makes it much easier for me to grade.
\[ [3] \quad f(a+h) - f(a) = (40 + 13h + h^2) - 40 = 13h + h^2 \]

\[ [4] \quad \text{DQ:} \quad \frac{f(a+h) - f(a)}{h} = \frac{13h + h^2}{h} = 13 + h \]

\[ [5] \quad \therefore \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} (13 + h) = 13. \]

And so

The slope of the line tangent to the graph of \( f(x) = x^2 + 3x \) at the point \( P(5, 40) \) is 13.

2. **Given**: \( f(x) = x^3 \) and \( a = 2 \).

**Required**: Find \( \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \) using the 5-Step Method, and interpret this result.

**Sol.**

\[ [1] \quad f(a) = f(2) = 8 \]

\[ [2] \quad f(a+h) = f(2+h) = (2+h)^3 = 1 \cdot 2^3 \cdot h^0 + 3 \cdot 2^2 \cdot h^1 + 3 \cdot 2^1 \cdot h^2 + 1 \cdot 2^0 \cdot h^3 \]
\[ = 8 + 12h + 6h^2 + h^3 \]

\[ [3] \quad f(a+h) - f(a) = (8 + 12h + 6h^2 + h^3) - 8 = 12h + 6h^2 + h^3 \]

\[ [4] \quad \text{DQ:} \quad \frac{f(a+h) - f(a)}{h} = \frac{12h + 6h^2 + h^3}{h} = 12 + 6h + h^2 \]

\[ [5] \quad \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} (12 + 6h + h^2) = 12 \]

And so

The slope of the line tangent to the graph of \( f(x) = x^3 \) at the point \( P(2, 8) \) is 12.
3. **Given**: \( f(x) = x^4 \) and \( a = -3 \).

**Required**: Find \( \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \) using the 5-Step Method, and interpret this result.

**Sol.**

[1]. \( f(a) = f(-3) = (-3)^4 = 81 \)

[2]. \( f(a+h) = f(-3+h) = (-3+h)^4 \)

\[
= 1(-3)^4 \cdot h^0 + 4(-3)^3 \cdot h^1 + 6(-3)^2 \cdot h^2 + 4(-3)^1 \cdot h^3 + 1(-3)^0 \cdot h^4
\]

\[
= 81 - 108h + 54h^2 - 12h^3 + h^4
\]

[3]. \( f(a+h) - f(a) = (81 - 108h + 54h^2 - 12h^3 + h^4) - 81 \)

\[
= -108h + 54h^2 - 12h^3 + h^4
\]

[4]. **DQ**: \( \frac{f(a+h) - f(a)}{h} = \frac{-108h + 54h^2 - 12h^3 + h^4}{h} = -108 + 54h - 12h^2 + h^3 \)

[5]. \( \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} (-108 + 54h - 12h^2 + h^3) = -108 \)

And so

The slope of the line tangent to the graph of \( f(x) = x^4 \) at the point \( P(-3, 81) \) is -108.

4. **Given**: \( f(x) = \sqrt{x} \) and \( a = 4 \).

**Required**: Find \( \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \) using the 5-Step Method, and interpret this result.

**Sol.**

[1]. \( f(a) = f(4) = \sqrt{4} = 2 \)

[2]. \( f(a+h) = f(4+h) = \sqrt{4+h} \)

[3]. \( f(a+h) - f(a) = \sqrt{4+h} - 2 \)

[4]. DQ:
\[
\frac{f(a+h)-f(a)}{h} = \frac{\sqrt{4+h} - 2}{h} = \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2}
\]
\[
= \frac{(4+h)-4}{h(\sqrt{4+h} + 2)} = \frac{h}{h(\sqrt{4+h} + 2)}
\]
\[
= \frac{1}{\sqrt{4+h} + 2}
\]

[5]. \[ \lim_{h \rightarrow 0} \frac{f(a+h)-f(a)}{h} = \lim_{h \rightarrow 0} \left( \frac{1}{\sqrt{4+h} + 2} \right) = \frac{1}{4} \]

And so

The slope of the line tangent to the graph of \( f(x) = \sqrt{x} \) at the point \( P(4, 2) \) is \( \frac{1}{4} \).

So, maybe now you can see that sometimes you do the “lion’s share” of the work in step (2), sometimes in step (3), and sometimes in step (4). It just depends upon the function involved.

******* ******* ******* ******* ******* ******* ******* ******* ******* ******* *******

OK, now let’s “up the ante” just a bit!

5. Given: \( f(x) = 2x^2 + 5x + 1 \) and \( a = 1 \).

Required: Find \( \lim_{h \rightarrow 0} \frac{f(a+h)-f(a)}{h} \) using the 5-Step Method, and interpret this result.

Sol. [1]. \( f(a) = f(1) = 2(1)^2 + 5(1)+1 = 8 \)

[2]. \( f(a+h) = f(1+h) = 2(1+h)^2 + 5(1+h)+1 \)
\[ = 2(1+2h+h^2)+5+5h+1 \]
\[ = 2 + 4h + 2h^2 + 5 + 5h + 1 = 8 + 9h + 2h^2 \]

[3]. \[ f(a+h) - f(a) = (8 + 9h + 2h^2) - 8 = \frac{9h + 2h^2}{h} \]

[4]. DQ: \[ \frac{f(a+h) - f(a)}{h} = \frac{9h + 2h^2}{h} = \frac{h(9 + 2h)}{h} = 9 + 2h \]

[5]. \[ \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} (9 + 2h) = 9 \]

And so

The slope of the line tangent to the graph of \( f(x) = 2x^2 + 5x + 1 \) at the point \( P(1, 8) \) is 9.

6. Given: \( f(x) = x^{\sqrt{x}} \) and \( a = 9 \).

Required: Find \( \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \) using the 5-Step Method, and interpret this result.

Sol. [1]. \( f(a) = f(9) = 9^{\sqrt{9}} = 27 \)

[2]. \( f(a+h) = f(9+h) = (9+h)^{\sqrt{9+h}} \)

[3]. \( f(a+h) - f(a) = (9+h)^{\sqrt{9+h}} - 27 \)

[Note: I could have started my "simplification" here, but I chose to wait until step 4.]

[4]. DQ:

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\[
\frac{f(a+h) - f(a)}{h} = \frac{(9+h)^\sqrt{9+h} - 27}{h} = \frac{1}{h}\left((9+h)^\sqrt{9+h} - 27\right)
\]

why?

= \frac{1}{h}\left((9+h)^\sqrt{9+h} - 9\cdot3\right)

why?

"hokey–pokey"

= \frac{1}{h}\left((9+h)^\sqrt{9+h} - (9+h)\cdot3 + (9+h)\cdot3 - 9\cdot3\right)

why?

= \frac{1}{h}\left\{\left((9+h)^\sqrt{9+h} - (9+h)\cdot3\right) + \left((9+h)\cdot3 - 9\cdot3\right)\right\}

why?

= \frac{1}{h}\left((9+h)^\sqrt{9+h} - 3\right) + 3\left((9+h) - 9\right)

why?

= \frac{1}{h}\left((9+h)^\sqrt{9+h} - 3\right) + 3h

ahaa! now I'm beginning to see!

= \left((9+h)^\sqrt{9+h} - 3\right) + \frac{3h}{h}

do the "conjugate"

= \left((9+h)^\sqrt{9+h} - 3, \frac{9+h}{h} + 3 \right) + \frac{3h}{h}

simplify

= \left((9+h)\left(\frac{(9+h) - 9}{h(\sqrt{9+h} + 3)}\right) + 3\right)

more

= \left((9+h)\left(\frac{h}{h(\sqrt{9+h} + 3)}\right) + 3\right)

= \left(\frac{1}{\sqrt{9+h} + 3} + 3\right) \text{ whew!!}

But, hey, it's just algebra!!!
And so

\[ \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \left( (9 + h) \left( \frac{1}{\sqrt{9 + h} + 3} \right) + 3 \right) \]

\[ = 9 \cdot \frac{1}{6} + 3 = \frac{3}{2} + 3 = \frac{9}{2} \]

And so

The slope of the line tangent to the graph of \( f(x) = x\sqrt{x} \) at the point \( P(9, 27) \) is \( \frac{9}{2} \).

Again we see that some problems take longer than other problems, but the 5 steps are always the same. So if you can avoid making careless errors in the steps, then you can get the problem right.

One final note about this problem – there is another way to do it. Just re-write the function as \( f(x) = \sqrt[3]{x^3} \), using the “laws of exponents,” and then applying the “5-Steps” to this function. In this case the “hokey-pokey” does not have to be applied, because the function is no longer the product of two functions, but the expansion may be a bit more difficult.

The techniques that we have studied so far are:

1. expanding a binomial (using, perhaps, Pascal’s triangle);
2. using the “conjugate” when there is a radical involved; and
3. dancing the “hokey-pokey” when the function is actually the product of two other functions.

There are more techniques, but these three will get us through a lot of limits. Now you just need to learn when to apply each and how to avoid making careless errors.

And so here is the “Practice Area” of this report.

PROBLEMS

INSTRUCTIONS: Using the “5-Step Method” find and interpret \( \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \)
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<th>#</th>
<th>Function and “a”</th>
<th>Partial Answer</th>
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<tbody>
<tr>
<td>1</td>
<td>$f(x) = x^2 - 2x$</td>
<td>$a = 3$ 4</td>
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<tr>
<td>2</td>
<td>$f(x) = \sqrt{x}$</td>
<td>$a = 1 \frac{1}{2}$</td>
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<tr>
<td>3</td>
<td>$f(x) = (x-1)^2 (x+2)^3$</td>
<td>$a = 2$ 176</td>
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<tr>
<td>4</td>
<td>$f(x) = x^3 - x^2 + 4x - 4$</td>
<td>$a = -1$ 9</td>
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Note: There are two ways to do problem #3 of this set.