§4.1: p. 232: #68. Show that 5 is a critical number of the function
\[ g(x) = 2 + (x-5)^3 \]
but \( g \) does not have a local extreme value at \( x = 5 \).

**Solution:**

1. Compute \( g'(x) = 3(x-5)^2 \)
2. Set \( g'(x) = 0 \) and solve: \( 3(x-5)^2 = 0 \) \( \Rightarrow x = 5 \)
   
   so \( x = 5 \) is a TYPE I CN.

Consider \( x_1 = 4.9 \) and \( x_2 = 5.1 \)

\[ g'(x_1) = g'(4.9) = 3(4.9-5)^2 > 0 \quad \text{and} \quad g'(x_2) = g'(5.1) = 3(5.1-5)^2 > 0 \]

\( \therefore \) The slope of the tangent line to the graph at points near the CN 5 is positive on both sides of the CN 5. Therefore \( g \) does not have a local extreme value at \( x = 5 \) (as per def. 2, p. 224).

**Alternative Method:**

\[ g(x_1) = g(4.9) = 2 + (4.9-5)^3 = 2 + (-0.1)^3 \]
\( \leq 2 \)

and \( g(x_2) = g(5.1) = 2 + (5.1-5)^3 > 2 \)

But \( g(5) = 2 \) \( \therefore \) \( g \) has no local extreme value at \( x = 5 \)!

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2. §4.3: p. 249: #50

Prove that for all \( x > 1 \),
\[ 2\sqrt{x} > 3 - \frac{1}{x} \]

**Try This:**

Define \( g(x) = 2\sqrt{x} - 3 + \frac{1}{x} \) for \( x > 1 \)

\[ g(1) = 0 \]
\[ g'(x) = 2 \cdot \frac{1}{2\sqrt{x}} - 0 - \frac{1}{x^2} \]

\[ = \frac{1}{\sqrt{x}} - \frac{1}{x^2} > 0 \]

\( x > 1 \)

1. \( x^2 > \sqrt{x} \)
2. \( \frac{1}{x^2} < \frac{1}{\sqrt{x}} \)
3. \( \therefore \) \( \frac{1}{\sqrt{x}} - \frac{1}{x^2} > 0 \)

\( \therefore \) The function \( g \) is always increasing but \( g(1) = 0 \) \( \therefore g(x) > 0 \) for all \( x > 1 \)

\[ 2\sqrt{x} - 3 + \frac{1}{x} > 0 \]
\[ 2\sqrt{x} > 3 - \frac{1}{x} \]
for all \( x > 1 \)
3. §4.3: p. 288: #3 Find two positive numbers whose product is 100 and whose sum is a minimum.

SoM:

1. Let the numbers be \( x \) and \( y \).

2. \( xy = 100 \) \( \therefore xy = 100 \)

3. Let \( f(x) \) be the sum of \( x \) and \( y \).

\[
\therefore f(x) = x + y \text{ but } y = \frac{100}{x}
\]

\[
\therefore f(x) = x + 100x^{-1}
\]

4. "MINIMIZE" the sum-function \( f(x) \).

   a. \( f'(x) = 1 - 100x^{-2} \)

   b. Set \( f'(x) = 0 \) \& Solve: \( 1 - \frac{100}{x^2} = 0 \)

\[ x = \pm 10 \]

(Throw out \( -10 \) b/c \( x \) must be pos.)

\[ \therefore x = 10 \]

5. Test for "min": (Use 2\textsuperscript{nd} Test) \( f''(x) = 200x^{-3} \) and \( f''(10) > 0 \) : \( x = 10 \) is a min. val.

\[ \therefore \] The two numbers are 10 and 10.

4. §5.2: p. 337: #9. Use MRule (n given)

to approx the integral. (4 dec. places)

\[
\int_{x=2}^{x=10} \sqrt{x^3 + 1} \, dx \quad n = 4.
\]

SoM:

(p. 332).

\[
\int_{x=a}^{x=b} f(x) \, dx \approx [f(x_1) + f(x_2) + \ldots + f(x_n)] \Delta x \quad (\ast)
\]

\[ \Delta x = \frac{b-a}{n} \] and \( x_i = \frac{i}{n} (x_1 + x_n) \)

\[ \Delta x = \frac{10-2}{4} = [2] \]

\[ \begin{array}{cccccc}
2 & 3 & 4 & 6 & 8 & 10 \\
\hline
x_i & 3 & 5 & 7 & 9 & \sqrt{750} \\
\hline
f(x_i) & \sqrt{28} & \sqrt{126} & \sqrt{344} & \sqrt{750} & \sqrt{750} \\
\hline
\end{array} \]

\[ \int_{x=2}^{x=10} \sqrt{x^3 + 1} \, dx \approx [\sqrt{28} + \sqrt{126} + \sqrt{344} + \sqrt{750}] \frac{2}{4} \approx 124.164 \text{ (4 dec. places)} \]

\[ \approx 124.1644 \text{ (4 dec. places)} \]
5. \( f(x) = \sqrt{x^2 + 1} - x \)

Use guidelines to sketch curve. (p.264)

**Solve**

1. **Domain:** (By inspection) \( \text{Dom}(y) : (-\infty, \infty) \)
2. **Intercepts:**
   - **X-int.** (when \( y = 0 \))
     \[ 0 = \sqrt{x^2 + 1} - x \quad \Rightarrow \quad \sqrt{x^2 + 1} = x \]
     \[ \text{Squ. b.s.} \quad x^2 + 1 = x^2 \quad \star \quad (\text{contradiction}) \]
     \[ \text{There are no X-intercepts.} \]
   - **Y-int.** (when \( x = 0 \))
     \[ y = \sqrt{0^2 + 1} - 0 = 1 \quad \text{Y-int is P}(0,1) \]
3. **Symmetry:** \( f(x) = \sqrt{x^2 + 1} - x \)
   - Consider \( f(-x) = \sqrt{(-x)^2 + 1} - (-x) = \sqrt{x^2 + 1} + x \)
   - So \( f(-x) \neq -f(x) \) (not odd)
   - \( f(-x) \neq f(x) \) (not even)
   - **No symmetry.**
4. **Asymptotes:** (Hor. Vert. Slant.)

**E.** Intervals of \( \neq \)

\[ f(x) = (x^2 + 1)^{1/2} - x \]
\[ f'(x) = \frac{1}{2}(x^2 + 1)^{-1/2} \cdot 2x - 1 = x(x^2 + 1)^{-1/2} - 1 \]

Set \( f'(x) = 0 \) \# **Solve**

\[ \frac{x}{(x^2 + 1)^{1/2}} = 1 \quad \Rightarrow \quad x = (x^2 + 1)^{1/2} \]

\# sq. b.s. \( x^2 = x^2 + 1 \) \#. Thus, there is no change in the increase/decrease of this function.

\( f(2) = \sqrt{5} - 2 \)
\( f'(a) = 1 \)
\( f'(a) > f'(x) \)
\[ \therefore f \text{ is decreasing on } (-\infty, \infty) \]
[E] \text{ Loc. Max/Min. } f \text{ has no LOCAL MAX, VAL, and NO LOCAL MIN, VAL } B < f'(x) < 0 \text{ for all } x \in \mathbb{R} \text{ (and there are no type II CN's.)}

[\text{Concavity}] \quad f(x) = \sqrt{x^2 + 1} - x

\begin{align*}
f'(x) &= x (x^2 + 1)^{-1/2} - 1 \\
\therefore f''(x) &= x \frac{d}{dx} [ (x^2 + 1)^{-1/2}] + 1 \cdot (x^2 + 1)^{-1/2} \\
&= \frac{-x^2}{(x^2 + 1)^{3/2}} + \frac{1}{(x^2 + 1)^{1/2} (x^2 + 1)} \\
&= \frac{-x^2 + x^2 + 1}{(x^2 + 1)^{3/2}} = \frac{1}{(x^2 + 1)^{3/2}} \\
\therefore f''(x) &= \frac{1}{(x^2 + 1)^{1/2}}
\end{align*}

\text{ and for any } x \in \mathbb{R} \quad f''(x) > 0

\therefore f \text{ is concave up on } (-\infty, \infty) \text{ and so there are no points of inflection.}

[E, §4.2, p. 239: #34] A number a is called a "fixed point" of a function f if f(x) = a.

\begin{proof}
If f(x) \neq 1 \text{ for all } x \in \mathbb{R}, \text{ the } f \text{ has at most 1 fixed point. (not more than 1)}
\end{proof}

\begin{proof}
\text{Assume not, i.e., assume there are more than 1 fixed points. (and hope for a contradiction.)}

\text{Assume there are 2 fixed points, a and b, and assume without loss of generality, that } a < b.

\text{Tacit hypothesis that } f \text{ is differentiable on } (-\infty, \infty),
\text{ and it follows that } f \text{ is continuous on } (-\infty, \infty)
\text{ b/c "differentiability implies continuity."

\therefore f \text{ is continuous on } [a, b]
\text{ and } f \text{ is differentiable on } (a, b)
\text{. \therefore } f \text{ satisfies the hypotheses of the MVT.
Thus, there must exist at least one number } c \in (a, b)
\text{ such that}
\begin{equation}
f'(c) = \frac{f(b) - f(a)}{b - a}
\end{equation}

"BUT" a + b \text{ are fixed pts. } \therefore f(a) = a \neq f(b) = b
\[ f'(x) = \frac{b-a}{b-a} = 1 \], but this contradicts the hypothesis \( f'(x) \neq 1 \) \( \forall x \in \mathbb{R} \).

1. There can't be 2 fixed points for this function.
2. There is at most one fixed point for \( f \).

4. Sec. 5.1: p. 270: #19
   Use Guidelines. \( y = x \sqrt{5-x} \)

   **Solv.**: \[ \text{Dom}(y) = (-\infty, 5] \]

   **B.** Intercepts:
   1. X-int \( (y = 0) \): \( 0 = x \sqrt{5-x} \)
      \( x = 0 \) and \( x = 5 \)
      \( P(0,0), P_2(5,0) \)
   2. Y-int \( (x = 0) \): \( P_1(0,0) \)

   **C.** Symmetry: \( f(x) = x \sqrt{5-x} \)
   \( f(-x) = (-x) \sqrt{5-(-x)} = -x \sqrt{5+x} \)

   \[ = -x \sqrt{5-x} \quad \text{(odd)} \]
   \[ = x \sqrt{5-x} \quad \text{(even)} \]

   \( f \) is neither odd nor even.

2. **ASYMPTOTES.** H/V/S.
   1. H.A. \[ \lim_{x \to \infty} x \sqrt{5-x} \quad \text{DNE.} \]
      \[ \lim_{x \to \infty} x \sqrt{5-x} = -\infty \]

      \[ \text{There are no H.A.} \]

   2. V.A. **None**

   3. S.A. **None**

**E.** Intervals \( \uparrow \downarrow \) : \( f(x) = x \sqrt{5-x} \)

   \[ f'(x) = x \cdot \frac{1}{2\sqrt{5-x}} \cdot (-1) + \sqrt{5-x} \]
   \[ = -\frac{x}{2\sqrt{5-x}} + \frac{\sqrt{5-x}}{1} \cdot \frac{2\sqrt{5-x}}{2\sqrt{5-x}} \]
   \[ = -\frac{x + 2(5-x)}{2\sqrt{5-x}} = \frac{10 - 3x}{2\sqrt{5-x}} \]

   \( \text{Dom} f' : (-\infty, 5) \)

   Set \( f'(x) = 0 \) \& Solve.

   \[ 10 - 3x = 0 \]
   \[ \Rightarrow x = \frac{10}{3} \quad (\frac{10}{3} \in \text{Dom } f) \]

   Try 2nd Der. Test:

   \[ f''(x) = \frac{1}{4(5-x)} \cdot \left[ 2\sqrt{5-x} \cdot (-3) - (10-3x) \cdot 2 \cdot \frac{1}{2\sqrt{5-x}} \right] \]
\[= \frac{1}{4(5-x)} \left[ (-6)\sqrt{5-x} + \frac{(10-3x)}{4\sqrt{5-x}} \right] \]
\[= \frac{1}{4(5-x)^{\frac{3}{2}}} \left[ -6\frac{(5-x)}{\sqrt{5-x}} + \frac{10-3x}{\sqrt{5-x}} \right] \]
\[= \frac{1}{4(5-x)^{\frac{3}{2}}} \left[ -6(5-x) + 10-3x \right] \]
\[= \frac{1}{4(5-x)^{\frac{3}{2}}} \left[ -30 + 6x + 10 - 3x \right] \]
\[= \frac{1}{4(5-x)^{\frac{3}{2}}} \left[ 3x - 20 \right] \]

Look at \( f'' \left( \frac{10}{3} \right) = \frac{1}{4\left( \frac{5}{3} \right)^{\frac{3}{2}}} \cdot (10 - 20) < 0 \)

\( f \) has a local max value of \( f \left( \frac{10}{3} \right) \) at \( x = \frac{10}{3} \)

\( f \) is increasing on \( (-\infty, \frac{10}{3}) \)

\( f \) is decreasing on \( (\frac{10}{3}, \infty) \)

\( f \) is CD on \( (-\infty, 5] \)

\[\text{Solve for } \theta. \]

\[\text{The indep. var. is } \theta, \text{ the angle.} \]

\[\text{The dep. var. is } F, \text{ the force.} \]

\[\text{Consider } \quad F(\theta) = \frac{\mu W}{\mu \sin(\theta) + \cos(\theta)} \]

\[\sin(\theta) = \frac{1}{\mu} \]

\[\cos(\theta) = \frac{\mu - 1}{\mu} \]

\[\text{Fnd } \theta \text{ dF} \text{ d} \theta, \text{ set it equal to zero, and solve for } \theta. \]

\[\text{F} \left( \theta \right) = \mu W \left( -\frac{\mu \cos(\theta) - \sin(\theta)}{\left( \mu \sin(\theta) + \cos(\theta) \right)^2} \right) \]

\[\text{Set } F(\theta) = 0 \]

\( \text{Numerator must be zero) i.e. } -\mu W \left[ \mu \cos(\theta) - \sin(\theta) \right] \]

\[\mu \cos(\theta) - \sin(\theta) = 0 \quad \therefore \mu \cos(\theta) = \sin(\theta) \]

\[\mu = \tan(\theta) \]