§12.3: P. 854: 15 — Direction angles & direction cosines. The direction angles, α, β, and γ of a vector
\[ \vec{V} = a\hat{i} + b\hat{j} + c\hat{k} \] are defined as follows:

α is the angle between \( \vec{V} \) and the positive x-axis (0 ≤ α ≤ π),

β is the angle between \( \vec{V} \) and the positive y-axis (0 ≤ β ≤ π),

γ is the angle between \( \vec{V} \) and the positive z-axis (0 ≤ γ ≤ π).

\[ \begin{align*}
\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma &= 1. \\
\text{These cosines are called the direction cosines of } \vec{V}. \\
\end{align*} \]

\[ \begin{align*}
&\text{[a] Show that } \cos \alpha = \frac{a}{\|V\|}, \cos \beta = \frac{b}{\|V\|}, \text{ and } \cos \gamma = \frac{c}{\|V\|} \\
&\text{and } \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1. \text{ [These cosines are called the direction cosines of } \vec{V}.] \\
&\text{[b] Unit vectors are built from direction cosines — Show that if } \\
&\vec{V} = a\hat{i} + b\hat{j} + c\hat{k} \text{ is a unit vector, then } a, b, \text{ and } c \\
&\text{are the direction cosines of } \vec{V}. \\
\end{align*} \]

\[ \begin{align*}
\text{[a] I shall show } \cos \alpha = \frac{a}{\|V\|} \text{ in detail. The other} \\
\text{two follow the same pattern.} \\
\text{[b] The } x\text{-axis and } \vec{V} \text{ determine a plane and } \alpha \text{ is} \\
\text{in that plane: } \vec{V} \text{. The } x\text{-axis} \\
\text{is parallel to the vector } \vec{c}, \text{ so} \\
\cos \alpha = \frac{\vec{c} \cdot \vec{V}}{\|\vec{c}\| \|\vec{V}\|} = \frac{\langle 1,0,0 \rangle \cdot \langle a, b, c \rangle}{\|\vec{V}\|} = \frac{a}{\|\vec{V}\|} \text{ so } \cos \alpha = \frac{a}{\|\vec{V}\|} \\
\text{And by analogous argument } \cos \beta = \frac{\vec{c} \cdot \vec{V}}{\|\vec{c}\| \|\vec{V}\|} = \frac{b}{\|\vec{V}\|} \text{ cont.} \\
\end{align*} \]
And $\cos \gamma = \frac{\mathbf{\hat{k}} \cdot \mathbf{\hat{V}}}{\| \mathbf{\hat{k}} \| \| \mathbf{\hat{V}} \|} = \frac{c}{\| \mathbf{\hat{V}} \|}$.

Thus $\cos \alpha = \frac{a}{\| \mathbf{\hat{V}} \|}$, $\cos \beta = \frac{b}{\| \mathbf{\hat{V}} \|}$, $\cos \gamma = \frac{c}{\| \mathbf{\hat{V}} \|}$.

One result provides a way to the next:

$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \left( \frac{a}{\| \mathbf{\hat{V}} \|} \right)^2 + \left( \frac{b}{\| \mathbf{\hat{V}} \|} \right)^2 + \left( \frac{c}{\| \mathbf{\hat{V}} \|} \right)^2$

$= \frac{a^2}{\| \mathbf{\hat{V}} \|^2} + \frac{b^2}{\| \mathbf{\hat{V}} \|^2} + \frac{c^2}{\| \mathbf{\hat{V}} \|^2} = \frac{a^2 + b^2 + c^2}{\| \mathbf{\hat{V}} \|^2}$

$= \frac{\mathbf{\hat{V}} \cdot \mathbf{\hat{V}}}{\| \mathbf{\hat{V}} \|^2} = \frac{\| \mathbf{\hat{V}} \|^2}{\| \mathbf{\hat{V}} \|^2} = 1$

Thus $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.

If $\mathbf{\hat{V}} = a\mathbf{\hat{i}} + b\mathbf{\hat{j}} + c\mathbf{\hat{k}}$ is a unit vector, then the direction cosines of $\mathbf{\hat{V}}$ are $a$, $b$, and $c$.

Proof: If $\mathbf{\hat{V}} = \langle a, b, c \rangle$ is a unit vector, then $\| \mathbf{\hat{V}} \| = 1$, so $\cos \alpha = \frac{a}{\| \mathbf{\hat{V}} \|} = a$, $\cos \beta = \frac{b}{\| \mathbf{\hat{V}} \|} = b$, and $\cos \gamma = \frac{c}{\| \mathbf{\hat{V}} \|} = c$.

Therefore, $\cos \alpha = a$, $\cos \beta = b$, and $\cos \gamma = c$. 

[Diagram and calculations]