§ 13.3 p. 920 # 19: The involute of a circle. If a string wound around a fixed circle is unwound while taut in the plane of the circle, its end P(x,y) traces an involute of the circle. In the accompanying figure, the circle in question is the circle $x^2 + y^2 = 1$ and the tracing point begins at (1,0). The unwound portion of the string is tangent to the circle @ Q, and $t$ is the radian measure of the angle from the positive x-axis to segment OQ. Derive the parametric equations

$$x = \cos t + ts\sin t, \quad y = \sin t - t\cos t, \quad t > 0$$

at the point P(x,y) for the involute.

1. Things to notice here:
   a. Coordinates of Q are $(\cos t, \sin t)$ since we have a unit circle.
   b. $t$ creates arclength $t$ which yields segment length $OQ = t$.

2. Slope of $\overline{OQ}$ is $\frac{\sin t}{\cos t}$ and $\overline{OP} \perp \overline{OQ}$, so slope of

   $\overline{OP}$ is $-\frac{\cos t}{\sin t}$

3. But slope of $\overline{OP}$ is $\frac{y - \sin t}{x - \cos t}$, so

   $$\frac{y - \sin t}{x - \cos t} = -\frac{\cos t}{\sin t}$$

4. Now considering vector $\overrightarrow{OP}$

   $$t = \| \overrightarrow{OP} \| = \sqrt{ (x - \cos t)^2 + (y - \sin t)^2 } = \sqrt{ (x - \cos t)^2 + (\frac{\cos t}{\sin t} (x - \cos t))^2 }$$

   $$= (x - \cos t) \sqrt{ 1 + \frac{\cos^2 t}{\sin^2 t} } = \frac{x - \cos t}{\sin t}$$

   So $t = \frac{x - \cos t}{\sin t}$ or $t \sin t = x - \cos t$.

5. The derivation for $y$ is left to the reader.

(\text{I'm sure there are other ways to do this, but this was the first thing that came to mind.)}