Some Thoughts on Lines and Planes (Relevant to §12.5)

1. We have seen that the vector equation for a line \( L \), through \( P_0(x_0, y_0, z_0) \) parallel to the vector \( \mathbf{v} = \langle a, b, c \rangle \) is
\[
\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \quad t \in \mathbb{R}
\]
where \( \mathbf{r}_0 = \overrightarrow{OP}_0 \).

This can also be written as a vector function
\[
\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \quad t \in \mathbb{R}.
\]

2. And if we look at components:
\[
\mathbf{r} = \langle x, y, z \rangle, \quad \mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle, \quad \mathbf{v} = \langle a, b, c \rangle
\]
\[
\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle
\]
\[
\therefore \langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle
\]

Now, two vectors are equal iff corresponding components are equal. So it follows that
\[
L: \begin{cases}
x = x_0 + ta \\
y = y_0 + tb \\
z = z_0 + tc
\end{cases} \quad t \in \mathbb{R}
\]
specifies the coordinates of a "generic" point on the line \( L \). These equations are called the parametric equations for \( L \).

3. Since there are lots of vectors (all parallel to \( \mathbf{v} \)) and lots of points \( P_0 \) (\( P_0 \) could be any specific point on \( L \)) which could be used to specify \( L \), it follows that there are lots of different-looking sets of parametric equations which could be used to describe \( L \).
Now, if we take our parametric equations for \( C \), solve each one for \( t \), and set them equal, we get the symmetric equations for \( C \):

\[
\begin{align*}
\dot{x} &= x_0 + ta & \Rightarrow & & t = \frac{x-x_0}{a} \\
\dot{y} &= y_0 + tb & \Rightarrow & & t = \frac{y-y_0}{b} \\
\dot{z} &= z_0 + tc & \Rightarrow & & t = \frac{z-z_0}{c} 
\end{align*}
\]

\[
\Rightarrow \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} \quad \text{provided} \quad a \neq 0 \land b \neq 0 \land c \neq 0
\]

II Distance from a Point \( S \) in space to a line \( C \) in space.

A We learned in class that the distance between a point \( S \) and a line \( C \) (parallel to a vector \( \vec{\n} \)) is given by

\[
d(S, C) = \frac{||\vec{0}S \times \vec{\n}||}{||\vec{\n}||}
\]

where \( P_0 \in \mathcal{L} \).

B Let us now recall why that is so.

First, look at the graphics:

\[
\begin{align*}
\vec{C} & \quad \vec{\n} \\
\vec{P_0} & \quad S
\end{align*}
\]

In vector form \( C \): \( \vec{r} = \vec{r}_0 + t\vec{\n} \) and \( \vec{r}_0 = \overrightarrow{OP_0} \) for \( P_0 \in \mathcal{L} \).

So,...

\[
\begin{align*}
P_0 & \quad \vec{P_0S} \\
S & \quad \vec{d} \quad \frac{\vec{d}}{||\vec{P_0S}||} = \sin \theta \Rightarrow d = ||\vec{P_0S}|| \sin \theta \\
C & \quad \vec{\n} \quad \frac{d}{||\vec{P_0S}||} \quad \frac{d}{||\vec{\n}||} \quad \frac{d}{||\vec{P_0S} \times \vec{\n}||} \quad \frac{d}{||\vec{\n}||} \quad \frac{d}{||\vec{P_0S} \times \vec{\n}||} \quad \frac{d}{||\vec{\n}||} \quad \frac{d}{||\vec{P_0S} \times \vec{\n}||} \\
\end{align*}
\]

\[
\therefore d(S, C) = \frac{||\vec{P_0S} \times \vec{\n}||}{||\vec{\n}||}, P_0 \in \mathcal{L}
\]

\( \text{cont.} \)
Now let's see how you do it. Assume you've been given $S$ and $C$ in parametric form.

1. If

$$C = \begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases} \quad t \in \mathbb{R}$$

Remember $P_0(x_0, y_0, z_0)$ is a point on the line, and $\vec{v} = \langle a, b, c \rangle$ is parallel to $C$.

2. Here's an example: Find the distance from $S(1, 2, 3)$ to the line $C$, given by $x = 2t$, $y = 1-t$, $z = -2+3t$, $t \in \mathbb{R}$.

Solve:

- a) $P_0(0, 1, -2)$

b) $\overrightarrow{PS} = \langle 1, 1, 5 \rangle$

c) $\vec{v} = \langle 2, -1, 3 \rangle$

d) $\overrightarrow{PS} \times \vec{v} = \langle 3+5, 10-3, -1-2 \rangle = \langle 8, 7, -3 \rangle$ or

$$\overrightarrow{PS} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 5 \\ 2 & -1 & 3 \end{vmatrix} = (3+5)\hat{i} - (3-10)\hat{j} + (-1-2)\hat{k} = 8\hat{i} + 7\hat{j} - 3\hat{k}$$

e) $\| \overrightarrow{PS} \times \vec{v} \| = (64 + 49 + 9)^{\frac{1}{2}} = (122)^{\frac{1}{2}}$

f) $\| \vec{v} \| = (4 + 1 + 9)^{\frac{1}{2}} = (14)^{\frac{1}{2}}$

g) $d(S, C) = \frac{\| \overrightarrow{PS} \times \vec{v} \|}{\| \vec{v} \|} = \frac{(122)^{\frac{1}{2}}}{(14)^{\frac{1}{2}}} = \left(\frac{122}{14}\right)^{\frac{1}{2}} = \left(\frac{61}{7}\right)^{\frac{1}{2}}$ exact.

h) $d(S, C) = \left(\frac{61}{7}\right)^{\frac{1}{2}}$ units (exact)