THEME: We have gone over the basic properties of vectors. But let me remind you that vectors have two distinguishing characteristics: magnitude and direction.

PART 1: We can isolate (and focus on) the magnitude of a vector by considering its “norm.” We symbolize this as follows: If \( \mathbf{a} \) is a vector, then its magnitude (or norm) is a real number which is indicated by \( \| \mathbf{a} \| \). An interesting (and deep) observation is that the process of “taking” or “finding” or “computing” the magnitude of a vector is a functional process or operation. That is,

\[
m(\mathbf{a}) = \| \mathbf{a} \| \tag{1.1}
\]

where \( m \) is the magnitude function, \( \mathbf{a} \) is an “element” of the vector space \( \mathcal{V} \) (\( \mathbf{a} \in \mathcal{V} \)) and the output value is a real number (\( \| \mathbf{a} \| \in \mathbb{R} \)). Another often-used way of showing such a functional relationship is

\[
\mathcal{V} \xrightarrow{\| \cdot \|} \mathbb{R}
\quad \mathbf{a} \xrightarrow{\| \cdot \|} \| \mathbf{a} \| \tag{1.2}
\]

indicating that the function \( \| \cdot \| \) “takes” a vector on the left and transforms it into the real number on the right.

One of the obvious, but very important properties of “norm” is this:

\[
\| \mathbf{a} \| = 0 \iff \mathbf{a} = \mathbf{0} \tag{1.3}
\]

That is, “the magnitude of a vector is zero if and only if the vector is the zero vector.”\(^2\)

There are three other “defining” properties of the magnitude or norm of a vector. You might as well internalize them (don’t just memorize them – internalize them!):

\begin{itemize}
  \item \( \forall \mathbf{a} \in \mathcal{V}, \; \| \mathbf{a} \| \geq 0. \)
  \item \( \forall \alpha \in \mathbb{R}, \forall \mathbf{a} \in \mathcal{V}, \; \| \alpha \mathbf{a} \| = |\alpha| \| \mathbf{a} \| \) where \( |\alpha| \) indicates the usual absolute value of the real number \( \alpha \), \( |\alpha| = \sqrt{\alpha^2} \).
  \item \( \forall \mathbf{a}, \mathbf{b} \in \mathcal{V}, \; \| \mathbf{a} + \mathbf{b} \| \leq \| \mathbf{a} \| + \| \mathbf{b} \|. \) (triangle inequality)
\end{itemize}

\(^1\) Note: We often write \( \mathbf{a} \in \mathcal{V} \) instead of \( \mathbf{a} \in \mathcal{V} \).

\(^2\) Please, please notice that there is a very important difference between the real number zero, 0, and the zero vector, \( \mathbf{0} \).
**PART 2:** Suppose that \( a \) is a “non-zero vector” \( (a \neq 0) \), then we can multiply \( a \) by the reciprocal of its magnitude. The result is a vector, we’ll call it \( u \). The important thing here is that the magnitude of \( u \) is 1. \( u \) is what we call a “unit vector.” The vector \( u \) focuses on and highlights the direction of the vector \( a \). The vector \( u \) is called “the direction of the vector \( a \).”

Let’s prove that if \( a \) is a non-zero vector, then \( \| u \| = 1 \).

**Proof:**

1. \( a \neq 0 \quad \therefore \quad \| a \| \neq 0 \quad \therefore \quad \frac{1}{\| a \|} \in \mathbb{R} \).

2. \( u = \frac{1}{\| a \|} a \quad \therefore \quad \| u \| = \frac{1}{\| a \|} \| a \| = \frac{1}{\| a \|} \| a \| = \frac{1}{\| a \|} = 1 \). \qed

How about a picture?

![Diagram of vectors]

**CONCLUSION:** Now also note the trivial observation that \( a = \| a \| u \). But what does this really mean? It means that we can break down any non-zero vector into its two defining characteristics – its magnitude, \( \| a \| \), and its direction, \( u \). And this really is not trivial; it is important.