Here is some practice at the important skill of verifying that a given function is a solution of a ODE. Verification is how you check your answers, and if you can’t check your answers, how do you know if they are right?

And since I’ve had some extra time on my hands during the short break between semesters, I’ve also included some review of important Calc II techniques we’ll be using in this course. Enjoy.

I. Verify that $\phi(t) = e^{-2t}$ is a solution of $y'' + 9y' + 14y = 0$.

Verification: [1] $\phi(t) = e^{-2t} \Rightarrow \phi'(t) = -2e^{-2t} \Rightarrow \phi''(t) = 4e^{-2t}$

[2]. Subs. [1] into the LHS of the ODE and simplify:

$$y'' + 9y' + 14y \overset{subs}{=} \phi'' + 9\phi' + 14\phi = (4e^{-2t}) + 9(-2e^{-2t}) + 14(e^{-2t})$$

$$= (4 - 18 + 14)e^{-2t} = 0$$

[3]. Thus, we see that the given function is in fact a solution to the ODE.

II. Verify that $\phi(t) = e^{t^2}$ is a solution of $\frac{dy}{dt} = 2ty$ on the half-plane $y > 0$, given that $y$ is a function of $t$.

Verification: [1]. $\phi(t) = e^{t^2} \Rightarrow \phi'(t) = 2te^{t^2}$

[2]. Subs. [1] into the LHS of the ODE and simplify: $\frac{dy}{dt} \overset{subs}{=} \frac{d\phi}{dt} = 2te^{t^2} = 2\phi$

[3]. Therefore, \[ \frac{d\phi}{dt} = 2t\phi \], and it follows (by def) that $\phi(t) = e^{t^2}$ is a solution of the given ODE.

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1 Let’s get this “arrow” notation straight right at the outset! Misuse of the “arrow” (⇒) is one of my pet-peeves! Technically, the arrow means “implies.” This means that the statement (or equation) that comes right before the arrow (the “antecedent”) is a sufficient condition, in-and-of itself, to imply the statement (or equation) that comes right after the arrow (the “consequent”). In writing math, we bend the rule just a little and use the arrow to mean that the antecedent plus possibly some other obvious facts together imply the consequent…. But in any case, the arrow means implies. It does not mean “equals.” (⇒) means “equals.” (⇒) means “implies.” Don’t confuse them!
III. Verify that \( \phi(t) = 3 \sinh(2t) \) is a solution of \( \frac{d^2 x}{dt^2} - 4x = 0 \).

**Verification:**

[1]. \( \phi(t) = 3 \sinh(2t) \Rightarrow \phi'(t) = 6 \cosh(2t) \Rightarrow \phi''(t) = 12 \sinh(2t) \)

[2]. Subs. [1] into the LHS of the ODE and simplify:

\[
\frac{d^2 x}{dt^2} - 4x \quad \text{subs} \Rightarrow \frac{d^2 \phi}{dt^2} - 4\phi = 12 \sinh(2t) - 4(3 \sinh(2t)) = 0
\]

[3]. Therefore, \( \frac{d^2 \phi}{dt^2} - 4\phi = 0 \), and it follows (by def) that \( \phi(t) = 3 \sinh(2t) \) is a solution of the given ODE.

IV. Here is an “INSIGHT” for you to consider. For example, in the ODE above, \( \frac{d^2 x}{dt^2} - 4x = 0 \), think of the “zero” as a function!

V. Now let’s complicate things just a bit. Up until now, all the solutions we have verified have been **explicit functions**. However, in many, many problems you will be solving in this course, the solutions will be expressed as **implicit functions**. So I need to show you what such solutions look like and the slightly different technique used for verification.

VI. Verify that \( xy - \ln(y) + c = 0 \) (where \( c \) is any constant) is a solution of \( \frac{dy}{dx} = \frac{y^2}{1-xy} \) on the half-plane \( y > 0 \).

**Verification:**

[1]. We are going to differentiate both sides of the alleged solution and use the result to “build” or “construct” the given ODE.

\[
xy - \ln(y) + c = 0
\]

\[
\frac{d}{dx}(xy - \ln(y) + c) = \frac{d}{dx}(0) \Rightarrow xy' + y - \frac{1}{y}y' = 0
\]

Now multiply both sides of this last equation above by \( y \) to “clear” the fraction, and then solve the resulting equation for \( y' \):
\[ y \left( xy' + y - \frac{1}{y} y' \right) = y \times 0 \Rightarrow xy' + y^2 - y' = 0 \]

\[ \Rightarrow (xy - 1)y' = -y^2 \Rightarrow y' = \frac{-y^2}{xy - 1} \Rightarrow y' = \frac{y^2}{1 - xy} \]

Now don’t you see that we have constructed the ODE from the alleged “solution.” Consequently, the given implicit function is indeed a solution to the ODE. ■

VII. More on Solutions – A solution to an ODE is a function, or, perhaps, a set of functions. Such a set of functions (or “set of solutions”) may consist of a family of functions and perhaps, one or more singular solutions.

If, for example, we have the ODE \( \frac{dy}{dx} = y \), then \( y_1 = e^x \) is a solution\(^2\). But watch this: let \( c \in \mathbb{R} \) be any real constant. Then I claim that \( y = ce^x \) is also a solution to the ODE\(^3\). Thus \( y = ce^x \) defines a 1-parameter family of solutions of the ODE \( y' = y \).

VIII. Example: Show that \( \phi(x) = c_1 e^x \cos(2x) + c_2 e^x \sin(2x) \) defines a 2-parameter family of solutions of
\[
y'' - 2y' + 5y = 0 \quad (*)
\]

Verification: [1].
\[
\phi = c_1 e^x \cos(2x) + c_2 e^x \sin(2x) = e^x (c_1 \cos(2x) + c_2 \sin(2x))
\Rightarrow \phi' = e^x (-2c_1 \sin(2x) + 2c_2 \cos(2x)) + e^x (c_1 \cos(2x) + c_2 \sin(2x))
\Rightarrow \phi' = e^x [(2c_2 + c_1) \cos(2x) + (c_2 - 2c_1) \sin(2x)]
\]

and it follows that
\[
\phi'' = e^x (-2 (2c_2 + c_1) \sin(2x) + 2 (c_2 - 2c_1) \cos(2x)) + e^x ((2c_2 + c_1) \cos(2x) + (c_2 - 2c_1) \sin(2x))
\Rightarrow \phi'' = e^x [(2c_2 - 2c_1) \cos(2x) + (2c_2 + c_1) \sin(2x)]
\Rightarrow \phi'' = e^x [(4c_2 - 3c_1) \cos(2x) + (-3c_2 - 4c_1) \sin(2x)]
\]

(Wow! Sometimes your “step [1]” can be a bear!

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\(^2\) I invite you to verify this!

\(^3\) Also verify this.
[2]. Now we substitute these values into (*):
\[ y'' - 2y' + 5y = 0 \quad \text{subs} \quad \phi'' - 2\phi' + 5\phi = 0 \]
\[ = \left\{ e^x \left[ \left( 4c_2 - 3c_1 \right) \cos(2x) + \left( -3c_2 - 4c_1 \right) \sin(2x) \right] \right\} \]
\[ - 2 \left\{ e^x \left[ \left( 2c_2 + c_1 \right) \cos(2x) + \left( c_2 - 2c_1 \right) \sin(2x) \right] \right\} \]
\[ + 5 \left\{ e^x \left[ c_1 \cos(2x) + c_2 \sin(2x) \right] \right\} \]
\[ = e^x \left[ 0 \cos(2x) + 0 \sin(2x) \right] = 0 \]

[3]. Thus we have shown that \( \phi'' - 2\phi' + 5\phi = 0 \), and it follows by definition that \( \phi(x) = c_1 e^x \cos(2x) + c_2 e^x \sin(2x) \) represents a family of solutions of (*). This family has two parameters, \( c_1 \) and \( c_2 \); therefore, the family is a 2-parameter family of solutions of (*). ■

IX. Example – An ODE With a Solution that Comprises a 1-Parameter Family and a Singular Solution:

The 1-parameter family defined by \( y = cx + c^2 \), \( c \in \mathbb{R} \) and the singular solution \( y = \frac{-1}{4} x^2 \) solve the (non-linear) ODE \( xy' - y + (y')^2 = 0 \).

X. Now Here Is A Nice Little Variation On Our Theme – Rewrite The ODE.

Show that \( y = x \tan(x) \) is a solution to \( xy' = y + x^2 + y^2 \) (\( * \))

Verification: [1]. Re-write (*) like this: \( xy' - y - x^2 - y^2 = 0 \) (\( ** \))

[2]. Then find the derivative of the alleged solution:
\[ y = x \tan(x) \quad \Rightarrow \quad y' = x \sec^2(x) + \tan(x) \]

[3]. Now substitute the results of [2] into the LHS of (**):

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5 The function \( y = \frac{-1}{4} x^2 \) is called “singular,” because (a) it is a solution and (b) there is no real number \( c \) such that \( y = \frac{-1}{4} x^2 \) is of the form \( y = cx + c^2 \).
\[ xy' - y - x^2 - y^2 \quad \text{subs} \quad x \left( x \sec^2(x) + \tan(x) \right) - (x \tan(x)) - x^2 - (x \tan(x))^2 \]
\[ = x^2 \sec^2(x) + x \tan(x) - x \tan(x) - x^2 - x^2 \tan^2(x) \]
\[ = x^2 \sec^2(x) - x^2 - x^2 \tan^2(x) \]
\[ = x^2 \sec^2(x) - x^2 \left( 1 + \tan^2(x) \right) \]
\[ = x^2 \sec^2(x) - x^2 \left( \sec^2(x) \right) = 0 \]

[4]. Thus \( y = x \tan(x) \) is by definition a solution to the ODE (**).

XI. As I have said many times before, functions describe processes. A differential equation may be thought of as a description of a process and the rate of change of the process and/or the rate of change of the rate of change of the process, and so forth. Functional descriptions of processes form a part of what we call mathematical modeling, and the formulation or creation of specific equations and/or differential equations is part of what is called the modeling process.

Let us stop for a moment to reflect on why we formulate such mathematical models in the first place. We create models in order to predict or anticipate the state of the (described) process under some different set of conditions.

I know this is a very general statement, so let me give you an example. If the model depends upon time as its independent variable and if it describes, let us say, the temperature of a solution at a given time, then the purpose and utility of the model is that it enables us to accurately predict or anticipate the temperature of the solution at some future time.

There’s a whole lot that goes into the process of modeling. We begin with what we know about the physical process, including what we know about the changes in the process over time and/or over space. In many cases these considerations lead to a differential equation or a system of differential equations, the solution of which constitutes at least a portion of the mathematical model.

Now it’s important to distinguish that “modeling the process” is different from a “model of the process.” Modeling the process is, for our purposes, an activity – it is the creation of a differential equation (or system of differential equations) which describes the way the physical process works. A model of the process is, for our purposes, the equation (or equations) which is (are) a solution of that differential equation (or system).

For simplicity of language, from here on out we’ll equate the concepts “modeling the
process” with “writing and solving an ODE,” and “model” with “solution to an ODE.” This may not be technically correct, but it will serve the purpose of this course.

XII. Now many times in this course we’ll be faced with an ODE which is an **Initial Value Problem** (IVP). In its simplest form an IVP will consist of a single ODE together with a set of **Initial Conditions** (IC). The solution to such an IVP is a specific function which (a) satisfies the ODE and (b) satisfies the IC.

XIII. **Example**: Solve this IVP: \((x^2 - 1)y' = 1\), \((*)\) and \(y(2) = 0\) \((**)\)

**Solution**: [1]. (This is the easy part!) \((x^2 - 1)y' = 1 \Rightarrow y' = \frac{1}{x^2 - 1}\) \((***)\)

[2]. Now it occurs to us that we can “get to” \(y\) from \(y'\) by “taking” the antiderivative, and so, following Jones’s Golden Rule of Equations\(^9\), we “take” the antiderivative of both sides of equation \((***)\):

\[
\therefore y = \int \frac{dx}{x^2 - 1} + C \quad (4*)
\]

[3]. Thus we have effectively solved for \(y\). But this solution is greatly in need of “simplification.” And this is why I picked this example at this time – Calc II is a prerequisite for this course for good reason; many of the Calc II “Techniques of Integration” are required to complete the solutions to ODEs. And now is a good time to begin our review of integration techniques. We could perform the integration on the RHS of \((4*)\) using a trig substitution, and I invite you to try it. I settled upon \(x = \csc \theta\) and it worked out\(^10\). However, the technique of choice here is “Partial Fractions.” It goes as follows:

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6 Please notice that we are now **solving** an ODE rather than simply **verifying** a given solution. We can do this because the solution simply requires finding an **antiderivative**, and we can all use “early practice” at finding antiderivatives!

7 The requirement imposed upon this problem by the IC can be thought of as this: Of all possible solutions to \((*)\) we want the one whose graph passes through the point \((2, 0)\).

8 Since our **ultimate goal** is to “solve for \(y\),” it is only natural to “solve for \(y'\) “ as a **preliminary step**.

9 **Jones’s Golden Rule of Equations**: Do unto one side of an equation just about anything you want, as long as you do the same unto the other side.

10 There are several different trig substitutions which will “work.” This just happens to be one which seemed to me to minimize the “paperwork” that followed.
\[
\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1) + B(x+1)}{(x+1)(x-1)} = \frac{Ax - A + Bx + B}{x^2 - 1}
\]

\[
\therefore \frac{1}{x^2-1} = \frac{Ax - A + Bx + B}{x^2 - 1} \quad (5*)
\]

I’ve purposefully put in more steps than are required or necessary, and I’ve done so in order to make it as clear as possible exactly how I got equation (5*).

Now, equation (5*) says that two rational expressions are equal and that they have “the same” denominator. Thus, it follows logically that their numerators must be equal, and we conclude that

\[
1 = Ax - A + Bx + B \quad (6*)
\]

But (6*) is a polynomial equation, so we’ll arrange it in standard order:

\[
1 = (A + B)x + (-A + B) \quad (7*)
\]

And now we’ll recall that two polynomials (the LHS and the RHS) are equal if and only if corresponding coefficients of the powers of x are equal. This consideration results in a system of linear equations in the unknowns A and B:

\[
\begin{align*}
A + B &= 0 \\
-A + B &= 1
\end{align*} \quad (8*)
\]

This system can be solved by elimination, substitution, Cramer’s rule, or other matrix methods. The easiest way in this particular problem is by Cramer’s rule, because I can “do it in my head,” getting \( A = -\frac{1}{2} \) and \( B = \frac{1}{2} \). But in case you’ve forgotten Cramer’s rule, or perhaps you didn’t get to it in Precalculus, you can solve for A and B using the elimination method. In either event, we ultimately get the result (substituting way back into the “second equation” in (5*))

\[
\frac{1}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1} = -\frac{1}{2} + \frac{1}{2} = 1 - \frac{1}{x+1} \quad (9*)
\]

[4]. Thus, equation (4*) can now be re-written as

\[
y = \frac{1}{2} \int \left[ \frac{1}{x-1} - \frac{1}{x+1} \right] dx + C = \frac{1}{2} \left[ \ln |x-1| - \ln |x+1| \right] + C = \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| + C
\]

\[
\therefore y = \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| + C \quad (10*)
\]
[5]. Well, that seemed to take a long time, and it did. But we are not yet finished. In case you've forgotten, this is an IVP, and so we must substitute the given x and y values into equation (10*) in order to determine the specific value of C:

\[
\begin{align*}
    y(2) &= 0 \quad \text{and} \quad y = \frac{1}{2} \ln \left| \frac{x - 1}{x + 1} \right| + C \\
    \Rightarrow 0 &= \frac{1}{2} \ln \left| \frac{2 - 1}{2 + 1} \right| + C = \frac{1}{2} \ln \left( \frac{1}{3} \right) + C \\
    &= \ln \left( \sqrt[3]{\frac{1}{3}} \right) + C = \ln \left( \frac{1}{\sqrt[3]{3}} \right) + C = \ln 1 - \ln \sqrt[3]{3} + C = -\ln \sqrt[3]{3} + C \\
    \therefore 0 &= -\ln \sqrt[3]{3} + C \\
    \therefore C &= \ln \sqrt[3]{3}
\end{align*}
\]

[6]. Thus our final answer\(^{11}\) is

\[
y = \frac{1}{2} \ln \left| \frac{x - 1}{x + 1} \right| + \ln \sqrt[3]{3} = \frac{1}{2} \ln \left| \frac{x - 1}{x + 1} \right| + \frac{1}{2} \ln 3 = \frac{1}{2} \ln 3 \left( \frac{x - 1}{x + 1} \right)
\]

\[
\therefore y = \frac{1}{2} \ln \left| 3 \left( \frac{x - 1}{x + 1} \right) \right| \quad \text{(final answer)}
\]

XIV. Here is another example requiring partial fractions\(^{12}\) (It is also an IVP): Solve

\[
(x + 1)(x^2 + 1)y' = 2x^2 + x \quad (\ast), \quad y(0) = 1 \quad (\ast\ast)
\]

[1]. Re-write the ODE: 

\[
y' = \frac{2x^2 + x}{(x + 1)(x^2 + 1)} \quad (\ast\ast\ast)
\]

---

\(^{11}\) Before I let the length of this problem frighten you terribly, let me say that once you understand what is going on in this type solution, these problems 'write-ups' go much faster. The steps are there to be done, but the lengthy explanations are not required. I would say that once you have figured out the pattern of solution, a problem like this should not take more that 6 to 8 minutes to write-up fairly neatly.

\(^{12}\) I'm going to work faster this time.
[2]. Decompose the RHS of (***) into partial fractions:

\[ \frac{2x^2+x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \]

\[ \Rightarrow 2x^2 + x = Ax^2 + A + Bx^2 + Cx + Bx + C = (A + B)x^2 + (B + C)x + (A + C) \]

\[ \Rightarrow A + B = 2, \quad B + C = 1, \quad A + C = 0 \]

And \( A + C = 0 \Rightarrow C = -A \). Now subs into the 2nd equation, and the first and second (modified) equations give us the system

\[ A + B = 2 \]
\[ -A + B = 1 \]

\[ \therefore A = \frac{1}{2}, \quad B = \frac{3}{2}, \quad \text{and} \quad C = -\frac{1}{2} \]

[3].

\[ \therefore \frac{2x^2+x}{(x+1)(x^2+1)} = \frac{1}{2} \left[ \frac{1}{x+1} + \frac{3x-1}{x^2+1} \right] \]

\[ \therefore y' = \frac{1}{2} \left[ \frac{1}{x+1} + \frac{3x-1}{x^2+1} \right] \]

and \[ y = \frac{1}{2} \int \left[ \frac{1}{x+1} + \frac{3x-1}{x^2+1} \right] dx + C = \frac{1}{2} \left\{ \int \frac{dx}{x+1} + 3 \int \frac{xdx}{x^2+1} - \int \frac{dx}{x^2+1} \right\} + C \]

\[ = \frac{1}{2} \left\{ \ln |x+1| + \frac{3}{2} \ln |x^2+1| - \arctan(x) \right\} + C \]

\[ = \ln \left[ (x+1)^{\frac{1}{2}} (x^2+1)^{\frac{3}{4}} \right] - \frac{1}{2} \arctan(x) + C \]
[4].

\[ y = \ln \left[ (x + 1)^{\frac{1}{3}} \left( x^2 + 1 \right)^{\frac{2}{3}} \right] - \frac{1}{2} \arctan(x) + C \]

[5]. Use IC to find C:

\[ 1 = \ln \left[ (0 + 1)^{\frac{1}{3}} \left( 0^2 + 1 \right)^{\frac{2}{3}} \right] - \frac{1}{2} \arctan(0) + C = \ln 1 - \frac{1}{2} (0) + C = 0 + 0 + C \]

\[ \therefore C = 1 \]

[6]. Thus, the final solution to the IVP is

\[ y = \ln \left[ (x + 1)^{\frac{1}{3}} \left( x^2 + 1 \right)^{\frac{2}{3}} \right] - \frac{1}{2} \arctan(x) + 1 \]