§14.3 Arc Length & Curvature.

We'll get to this in a minute.

Back up to §14.2: p 893. Tangent Vector.

If \( \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle \) is a vector function, then \( \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \) is the tangent vector at \( t \).

Of GREAT importance to us is the unit tangent vector (function), which we shall call the Principal Unit Tangent Vector (function).

PUT vector.

Computing the PUT vector (function).

\[
\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \quad (p 893)
\]

Modified § 14.2: p 897: # 21

Given: \( \mathbf{r}(t) = \langle t, t^2, e^t \rangle \)

Required: \( \mathbf{T}(t), \mathbf{T}'(t), \mathbf{T}''(t), \) and \( \mathbf{T}(t) \times \mathbf{T}''(t), \).

Soln. \[
\mathbf{T}(t) = \frac{1}{\sqrt{1 + 4t^2 + 9t^4}} \langle 1, 2t, 3t^2 \rangle
\]

Find \( \mathbf{T}(t) \).

1. \( \|\mathbf{T}'(t)\| = \sqrt{1 + 4t^2 + 9t^4} \)

2. \[
\mathbf{T}'(t) = \frac{1}{\sqrt{1 + 4t^2 + 9t^4}} \langle 4t, 2, 6t^2 \rangle
\]

3. \[
\mathbf{T}''(t) = \frac{1}{\sqrt{1 + 4t^2 + 9t^4}} \left( \frac{24t^2 - 6t^2}{\sqrt{1 + 4t^2 + 9t^4}} \right)
\]

4. \[
\mathbf{T}'(t) \times \mathbf{T}''(t) = \langle 1, 2t, 3t^2 \rangle \times \langle 0, 2, 6t \rangle
= \langle 12t^2 - 6t^2, 0 - 6t, 2 - 0 \rangle = \langle 6t^2, -6t, 2 \rangle
\]

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In Calc 2

\[
L = \int_{t=a}^{t=b} \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt
\]
\[ L = \int_{t=a}^{t=b} \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} \, dt. \]

If \( \vec{r}(t) = \langle f(t), g(t), h(t) \rangle = \langle x(t), y(t), z(t) \rangle \)
then the function \( \vec{r} \) traces a space curve.

\[ \vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle \]

and \( \| \vec{r}'(t) \| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}^{1/2} \)

\[ \text{By Subs} \]

\[ L = \int_{t=a}^{t=b} \| \vec{r}'(t) \| \, dt \]

\[ D = RT \]

\[ \text{AFTER CLASS} \]

\[ \text{III Worked Example -- 5.14.3 p. 204; #1} \]

Find the length of the curve. \( \vec{r}(t) = \langle 2\sin t, 5t, 2\cos t \rangle, \]
\(-10 \leq t \leq 10.\)

\[ \text{Sol} \]

This space curve is a spiral helix w/ y-axis its axis.

2. \( \vec{r}'(t) = \langle 2\cos t, 5, -2\sin t \rangle \)

3. \( \| \vec{r}'(t) \| = \sqrt{4\cos^2(t) + 25 + 4\sin^2(t)}^{1/2} = \sqrt{29}^{1/2} \]

\[ L = \int_{t=-10}^{t=10} \| \vec{r}'(t) \| \, dt \] is the formula.

\[ L = \int_{t=-10}^{t=10} 29^{1/2} \, dt = 29^{1/2} \left[ t \right]_{t=-10}^{t=10} = 29^{1/2} \left[ 10 - (-10) \right] = 20 \cdot 29^{1/2} \text{ linear units.} \]

\[ L = 20 \cdot 29^{1/2} \text{ units.} \]