Last Topic in CHAIN RULE, § 14.4—

IMPLICIT DIFFERENTIATION — p. 985.

(A) $F(x,y)$ is differentiable.

(2) The equation $F(x,y) = 0$ defines $y$ IMPLICITLY as a (differentiable) function of $x$. [Say $y = h(x).$]

$x^2 + y^2 \quad \ldots \quad z = F(x,y) = x^2 + y^2$

Hummm. $F(x,y) = x^2 + y^2 - 4$

Set $F(x,y) = 0: \quad x^2 + y^2 - 4 = 0$

Defines TWO functions implicitly:

$y_1 = \sqrt{4 - x^2}$ and $y_2 = -\sqrt{4 - x^2}$

Back to the Implicit Diff. — $z = F(x,y) = 0$

$$\frac{dz}{dx} = \frac{\partial F}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx}$$

But $z = 0$ so $\frac{dz}{dx} = 0$

$$\therefore \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} \quad \text{or} \quad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

Very Useful.

\[ xy + y^2 - 3x - 3 = 0 \quad \text{at} \quad P(1,1) \]

Assume (*) defines \( y \) as a diff. fun. of \( x \) and find \( \frac{dy}{dx} \bigg|_{(1,1)}. \)

**Solution**

1. \( F(x,y) = xy + y^2 - 3x - 3 \)

2. \( F_x = y - 3, \quad F_y = x + 2y \)

3. \( \frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-(y-3)}{x+2y} = \frac{3-y}{x+2y} \)

4. \( \frac{dy}{dx} \bigg|_{(1,1)} = \frac{3-1}{1+2} = \frac{2}{3} \)

§ 14.5: Directional Derivatives & Gradient Vector.

(p. 989)

A. I need a direction (vector of mag. 1)
and I need a derivative.
And to have a derivative, I need a function.
\( z = f(x,y) \) \( \implies \) \( \mathbf{u} = u_1 \hat{i} + u_2 \hat{j} = \langle u_1, u_2 \rangle \)

We define the symbol \((\frac{df}{ds})_{\mathbf{u}, P_0}\) as.

\[ \left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s} \]

Alternative symbolism

\( (\nabla f)_{P_0} \)
Recall \[ \frac{\partial f}{\partial x} \bigg|_{P_0(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \]

Compare with
\[ (D_u f)_{P_0} = \lim_{s \to 0} \frac{f(x_0 + u_1 s, y_0 + u_2 s) - f(x_0, y_0)}{s} \]

Note
\[ (D_\xi f)_{P_0} = \lim_{s \to 0} \frac{f(x_0 + 1 s, y_0 + 0 s) - f(x_0, y_0)}{s} \]
\[ \xi = (1, 0) \]

simplify
\[ = \lim_{s \to 0} \frac{f(x_0 + s, y_0) - f(x_0, y_0)}{s} \]

Do you see the connection?