We discussed open connected regions and simple, positively oriented closed curves.

A problem from §16.2.

§16.2: p. 1143: # 25  Find the circulation and the flux.  \( \mathbf{F} = \langle x, y \rangle \)

around and across the closed path \( C_1 U C_2 \)

\( C_1: \mathbf{F}_1(t) = \langle a \cos t, a \sin t \rangle \quad 0 \leq t \leq \pi \)

\( C_2: \mathbf{F}_2(t) = \langle t, 0 \rangle \quad -a \leq t \leq a \)

Solve

2  Circulation

a  \( C_1: \mathbf{F} = \langle a \cos t, a \sin t \rangle, \quad C_2: \mathbf{F} = \langle t, 0 \rangle \)

b  Circ: \[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \]

\[ = \int_{t=0}^{t=\pi} \langle a \cos t, a \sin t \rangle \cdot \langle -a \sin t, a \cos t \rangle \, dt \]

\[ + \int_{t=-a}^{t=a} \langle t, 0 \rangle \cdot \langle 1, 0 \rangle \, dt \]

\[ = \int_{t=0}^{t=\pi} (-a^2 \cos t \sin t + a^2 \sin t \cos t) \, dt + \int_{t=-a}^{t=a} t \, dt \]

\[ = 0 + \left[ \frac{t^2}{2} \right]_{t=-a}^{t=a} = 0 + 0 = 0 \]

C: The circulation is 0. along \( C_1 \).
3 Flux. The flux integral is \( \int_C \vec{F} \cdot d\vec{n} \, dt \) but the working formula (p. 1141) is

\[
\int_C (M \, dy - N \, dx)
\]

\( \vec{F} = \langle M, N \rangle \)

\[\text{on } C_1: \vec{r}_1 = \langle \cos t, \sin t \rangle \]

\[
M = \cos t \quad N = \sin t \\
dx = -\sin t \, dt \\
dy = \cos t \, dt
\]

\[\text{on } C_2: \vec{r}_2 = \langle t, 0 \rangle \]

\[
M = t \\
N = 0 \\
dx = dt \\
dy = 0
\]

\[\text{Flux: } \int_C M \, dy - N \, dx = \int_{C_1} M \, dy - N \, dx + \int_{C_2} M \, dy - N \, dx
\]

\[
= \int_{t=0}^{t=\pi} (\cos t \cos t + \sin t \sin t) \, dt \\
+ \int_{t=-\pi}^{t=\pi} t \, dt - 0 \, dt
\]

\[
= a^2 \int_{t=0}^{t=\pi} 1 \, dt + 0 = \boxed{a^2 \pi}
\]

\[\text{The flux is } a^2 \pi \text{ outward.}\]

**A** Hypotheses: \( \vec{F} = \langle M, N, P \rangle \) \( M, N, P \) are continuous throughout an open region \( D \) in 3-space.

Then \( \exists \ f \) such that \( \nabla f = \vec{F} \) \( \iff \) for all points \( A, B \in D \)

\[ \int_{A}^{B} \vec{F} \cdot d\vec{r} \] is independent of Path.

And if the integral is independent of path, then

\[ \int_{A}^{B} \vec{F} \cdot d\vec{r} = f(B) - f(A). \]


\[ \int_{(0,0,0)}^{(2,3,-6)} 2xdx + 2ydy + 2zd\!dz \]

**Sol:**

Is \( 2xdx + 2ydy + 2zd\!dz \) exact.
Which means is \( \vec{F} = \langle 2x, 2y, 2z \rangle \) conservative.

\[ \frac{\partial M}{\partial y} = 0 \quad \frac{\partial N}{\partial x} = 0 \quad \checkmark \]
\[ \frac{\partial M}{\partial z} = 0 \quad \frac{\partial P}{\partial x} = 0 \quad \checkmark \]
\[ \frac{\partial N}{\partial z} = 0 \quad \frac{\partial P}{\partial y} = 0 \quad \checkmark \]

\[ \therefore \vec{F} \text{ is conservative.} \]
\[ \therefore \exists \text{ a potential function, } f \text{ such that } \nabla f = \vec{F} \]
After-Class Continuation...

2. \[ f(x, y, z) = \int 2x \, dx = x^2 + g(y, z) \] (*)

b. By theory \( N = \frac{\partial f}{\partial y} \). By the above \( \frac{\partial f}{\partial y} = 0 + \frac{\partial g}{\partial y} \).

By the "problem" \( N = 2y \)

\[ \therefore \frac{\partial g}{\partial y} = 2y \quad \text{Thus} \quad g(y, z) = y^2 + g_1(z) \] (**) 

And combining (*) and (**) we get 

\[ f(x, y, z) = x^2 + y^2 + g_1(z) \] (***)

c. By theory \( P = \frac{\partial f}{\partial z} \). By (***), \[ \frac{\partial f}{\partial z} = \frac{\partial g_1}{\partial z} \, . \]

By the "problem" \( P = 2z \) \[ \therefore \frac{\partial g_1}{\partial z} = 2z \quad \text{and} \quad \text{it follows that} \quad g_1(z) = z^2 + C \] (x).

d. So... \[ f(x, y, z) = x^2 + y^2 + z^2 + C \]

3. By FTLI if \( A(0, 0, 0) \) \& \( B(2, 3, -6) \) 

\[ \int_A^B 2x \, dx + 2y \, dy + 2z \, dz = f(B) - f(A) = 2^2 + 3^2 + (-6)^2 - 0 \]

\[ = 4 + 9 + 36 = 49 \]