54.1.3: Solutions of Linear Equations (p. 120)

A. Homogeneous \[ u'' = 0 \]
   Non homogeneous \[ u'' = q(x) \]

B. Superposition Principle for Homogeneous Equations

   Example: If \[ y'' + P(x)y' + Q(x)y = 0 \] (\(*\))
   and if \( y_1 \) is a solution to \((*)\) and if
   \[ y_2 = \ldots \] (then)
   \[ y = c_1 y_1 + c_2 y_2 \]
   is a solution to \((*)\)
   where \( c_1, c_2 \) are any constants whatsoever.

C. FACT: A linear homogeneous ODE always has \( y = 0 \) as a solution.

D. Example: Consider \[ y'' + 4y = 0 \] (\(\ast\ast\))
   a) Claim \( y_1 = \cos(2x) \) is a sol. to \((\ast\ast)\)
   Verification:
      1. \( y_1' = -2\sin(2x) \) \(\land\) \( y_1'' = -4\cos(2x) \)
      2. "Plug"
      \[ y_1'' + 4y_1 = -4\cos(2x) + 4(\cos(2x)) = 0 \]
   b) Claim: \( y_2 = \sin(2x) \) is a sol. to \((\ast\ast)\)
   Verification is similar to that above.

   c) Here's the point of this example:
      \( y_3 = 2\cos(2x) - 7\sin(2x) \) is also a sol. to \((\ast\ast)\)
Criterion (one condition) to guarantee L.I. solutions to a given $n^{th}$-order LHODE.

$y_1, y_2, \ldots, y_n$ are solutions to

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

on $I = [a, b]$ or $I = (a, b)$

Then

The set \{ $y_1, y_2, \ldots, y_n$ \} is a L.I. set.

$\iff$ \( W(y_1, \ldots, y_n)(x) \neq 0, \forall x \in I \).

\[ \square \] A set of $n$ Lin. Indep. solutions to an $n^{th}$-order LHODE is called a fundamental set of solutions. (DEFN)

\[ \square \] A fund. set forms a BASIS for the solution space of the LHODE, i.e. any soln can be written as

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

for some constants $c_i$, $i = 1, \ldots, n$.

GENERAL SOLUTION (GENSOL)