More from §4.1.3 - p. 125
Non-Homogeneous LODE (w/ const. coeff.)

Simple Example: \( y'' + 5y' + 6y = \sin(x) \) (**)

1. To solve (**), we first solve the related homogeneous equation (complementary equation)
   \[ y'' + 5y' + 6y = 0, \]
   getting (via simple-minded methods to be taught in the near future) a complementary solution
   \[ y_c = c_1 e^{-2x} + c_2 e^{-3x} \] ← Gen Sol to homog eq.

2. Then we must find (somehow) a particular solution to (**), call it \( y_p \)

3. Form Gen Sol to (**): \( y = y_c + y_p \)

Read pages 125-128 for examples.

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§4.2: Constructing a Second Solution from a Known Solution (p. 133)

Method RO (Reduction of Order)
Requires 0 or above LODEs

2. One Solution

3. Use RO to reduce order

4. Solve reduced Eq.

5. Form Gen Sol to original problem by Superposition.
We use a subs similar to that of "Homog. Eq. in the sense of Euler, ChZ."

Suppose that we have

\[ y'' + 5y' + 6y = 0 \quad (\star) \]

Claim is that \( y_1 = e^{-2x} \) is a sol. to \((\star)\)

Aside: Verification.

\[
y_1 = e^{-2x}, \quad y_1' = -2e^{-2x}, \quad y_1'' = 4e^{-2x}
\]

Now P\&C.

\[
y_1'' + 5y_1' + 6y_1 = (4e^{-2x}) + 5(-2e^{-2x}) + 6(e^{-2x})
\]

\[ = e^{-2x} \left[ 4 - 10 + 6 \right] = 0 \quad \text{check} \]

Now our real problem for this day is use \( y_1 \) to find another sol.

Technique: Let \( y_2 = u(x)y_1 = y_2 = ue^{-2x} \)

where the assumption is that there exists some function \( u(x) \)

such that \( y_2 = ue^{-2x} \) is a solution to \((\star)\).

\[
y_2' = u(-2e^{-2x}) + u'e^{-2x} = (u' - 2u)e^{-2x}
\]

\[
y_2'' = (u' - 2u)(-2e^{-2x}) + (u'' - 2u')e^{-2x}
\]

\[ = (u'' - 4u' + 4u)e^{-2x} \]

Now subs: This is our assumption (above) that \( y_2 \) is a sol. to \((\star)\)

\[
0 = y_2'' + 5y_2' + 6y_2 = [(u'' - 4u' + 4u)e^{-2x}] + 5[(u' - 2u)e^{-2x}] + 6ue^{-2x} = \text{cont}...
\]
\[ (u'' - 4u' + 4u) + 5(u' - 2u) + 6u \} e^{-2x} = 0 \]

\[ \{ u'' - 4u' + 4u + 5u' - 10u + 6u \} e^{-2x} = 0 \]

\[ \{ u'' + u' \} e^{-2x} = 0 \]

4. **Summarizing the calculations above, we have**

\[ (u'' + u') e^{-2x} = 0 \]

5. **Observe that** \( e^{-2x} \) **is never zero, so**

\[ u'' + u' = 0 \] (**)  

6. **To facilitate our calculations, let** \( w = u' \). **Then (**) becomes**

\[ (w' + w = 0) (***) \]

and this is a first order ODE. — which we can solve via the methods of Ch 2 (integrating factor).

7. **Let** \( \mu(x) = \exp \left( \int 1 \, dx \right) = e^x \) **and “apply it” to (**)**

\[ e^x (w' + w) = e^x (0) \]

\[ (e^x w)' = 0 \]  

\[ e^x w = C \]

\[ w = Ce^{-x} \]  

Integrate.

8. **Now “un-sub”** \( w = Ce^{-x} \) **and** \( w = u' \)

\[ u' = Ce^{-x} \]

\[ u = C \int e^{-x} \, dx = -Ce^{-x} + c \]

9. **cont...**
Now we go back to $y_2 = ce^{-2x}$ and "un-sub"

$$y_2 = (c_1 - ce^{-x})e^{-2x} = ce^{-2x} - ce^{-3x}$$

We want a specific solution and we want it to be independent of $e^{-2x}$, so pick $c_1 = 0$ and $c = -1$

$$y_2 = e^{-3x}$$

This is a second solution to $(*)$ $y'' + 5y' + 6y = 0.$