§ 5.2: Damped Motion, p. 192

A § 5.2; p. 198: #7. 4-lb wt. \( k = 2 \) lb/ft

\[ \beta = 1 \frac{\text{lb}\cdot\text{sec}}{\text{ft}} \]
\[ x_0 = -1 \text{ ft} \quad v_0 = 8 \text{ ft/sec} \]

Required:

A Time at which wt. passes through equil. position.

B Time at which wt. attains extreme displacement from equil. position

C What is the position of wt. at time B?

Solution:

\[ mx'' + \beta x' + kx = 0 \] (Ⅰ)

a. Use \( g = 32 \) ft/sec² \( w = mg \) \( i.e., m = \frac{w}{g} = \frac{4}{32} \)

\( \therefore m = \frac{1}{8} \) slugs.

\( \therefore (Ⅰ) \) becomes \( \frac{1}{8} x'' + x' + 2x = 0 \)

\( \therefore x'' + 8x' + 16x = 0 \)

\( 2\lambda = 8 \quad \lambda = 4 \)
\( \omega^2 = 16 \quad \omega = 4 \)

Solution:

\[ m^2 + 8m + 16 = 0 \]

\( (m+4)^2 = 0 \) (critically damped)

\[ x(t) = c_1 e^{-4t} + c_2 te^{-4t} \]

i. \( x' = -4c_1 e^{-4t} + c_2 (-4te^{-4t} + e^{-4t}) \)

ii. \( -1 = x(0) = c_1 \quad c_1 = -1 \)

iii. \( 8 = x'(0) = -4c_1 + c_2 \quad \rightarrow 4 + c_2 \)

\[ c_2 = 4 \]
\[ x(t) = -e^{-4t} + 4te^{-4t} \]
\[ x(t) = e^{-4t} (4t - 1) \]
\[ (*) \]

2. **Graph**

3. To solve \( A \). The wt. passes through equilibrium pos. when \( x = 0 \). So set \( (*) \) equal to zero & solve.

\[ 0 = e^{-4t} (4t - 1) \quad \Rightarrow \quad t = \frac{1}{4} \text{ sec.} \]

To solve \( B \) (time of extreme displacement).
Set \( x' = 0 \) & solve.

\[ 0 = x' = 4e^{-4t} + (-4)e^{-4t} (4t - 1) \]
\[ = [4 - 4(4t - 1)] e^{-4t} \]
\[ = (8 - 16t) e^{-4t} \quad \therefore \quad t = \frac{1}{2} \text{ sec} \]

To solve \( C \) (position at time \( B \)):

\[ x\left(\frac{1}{2}\right) = e^{-2} \approx 0.135335 \text{ ft.} \]

4. **Summary**: The wt. passes through the equilibrium position @ \( t = \frac{1}{4} \text{ sec.} \). The wt. has its extreme displacement (of \(-1 \text{ ft.}\)) at \( t = 0 \text{ sec.} \), and its extreme pos. displacement of approx. \( 0.135 \text{ ft.} \) \((\frac{1}{2}e^2 \text{ ft. exact})\) at time \( t = \frac{1}{2} \text{ sec.} \).