II Quiz: 2.5: $\int \ln x \, dx$

Hint: $\ln x = 1 \cdot \ln x$

Solution

1. $\int \frac{d}{dx} \ln x \, dx = \frac{1}{x}$
   $\int \frac{1}{x} \, dx = \ln x + C$

2. $\int \ln x - x + C = x \ln x - x + C$

3. Check: $\frac{d}{dx} \left[ x \ln x - x + C \right] = x \cdot \frac{1}{x} + 1 \cdot \ln x - 1 + 0$
   $= 1 + \ln x - 1 = \ln x$

II Integrands - PFDs (Variations on a THEME!)

4. \( \frac{x^2 + 5x + 5}{x^2 + 3x + 2} \)

Solution

1. \( x^2 + 3x + 2 \left( \frac{x^2 + 5x + 5}{x^2 + 3x + 2} \right) \)
   \( = \frac{1}{2x + 3} \)

2. \( \frac{x^2 + 5x + 5}{x^2 + 3x + 2} = 1 + \frac{2x + 3}{x^2 + 3x + 2} \)

3. Concentrate on \( \frac{2x + 3}{x^2 + 3x + 2} = \frac{2x + 3}{(x + 2)(x + 1)} = \frac{1}{x + 2} + \frac{1}{x + 1} \)

4. Quick Check: \( \frac{1}{x+1} \cdot \frac{1}{x+2} = \frac{2x + 3}{(x + 2)(x + 1)} \)

5. \( \frac{x^2 + 5x + 5}{x^2 + 3x + 2} = 1 + \frac{1}{x + 2} + \frac{1}{x + 1} \)
\[ PFD \quad \frac{1}{x^3 + 2x^2 + x} \]

Solution

1. \[ \frac{1}{x (x^2 + 2x + 1)} = \frac{1}{x (x+1)(x+1)} \]

2. \[ 1 = A(x+1)^2 + Bx(x+1) + Cx \]

3. \[ \begin{cases} 
A + B = 0 \\
2A + B + C = 0 \\
A = 1 
\end{cases} \]

\[ \begin{array}{c}
A = 1 \\
B = -1 \\
C = -1
\end{array} \]

4. \[ \frac{1}{x^3 + 2x^2 + x} = \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2} \]

5. Q.C. LCD of RHS \[ \frac{(x+1)^2 - x(x+1) - x}{x(x+1)^2} = \frac{x^2 + 2x + 1 - x^2 - x - x}{x(x+1)^2} \]
Given
\[ \frac{x}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1} \]

Set up.

\[ \begin{align*}
& \text{Back together} \\
& x = A(x^2+x+1) + (x+1)(Bx+C) \\
& x = Ax^2 + Ax + A + Bx^2 + Cx + Bx + C \\
& x = (A+B)x^2 + (A+B+C)x + (A+C)
\end{align*} \]

3. \[ \begin{cases} 
A+B = 0 & (1) \\
A+B+C = 1 & (2) \\
A+C = 0 & (3)
\end{cases} \]

Bring your calculator to class tomorrow and I'll show you how to do this part w/ matrices & determinants.

End of class ☑️ after class ☑️

I'm going to finish this problem the "long way."

4. Notice that from eqs. (1) & (3) we get \( B = -A \) and \( C = -A \). Subs.
these results into eq. (2): \( A - A - A = 1 \), \(-A = 1\), \( A = -1 \).

Then "back-sub" into eq's (4) and (5): \( B = 1 \), \( C = 1 \).

5. \[
\frac{x}{(x+1)(x^2+x+1)} = \frac{x+1}{x^2+x+1} - \frac{1}{x+1}
\]
Q.C. (Quick Check) \( \frac{A}{B} + \frac{C}{D} = \frac{AD + BC}{BD} \). We check the "AD + BC"
from the RHS of (*)

\[(x+1)(x+1) - (x^2 + x + 1)\]
\[= x^2 + 2x + 1 - x^2 - x - 1\]
\[= x \leftarrow \text{Checks!} \]

Just to finish this last one out: (I can't Resist!)

Integrate \( \int \frac{x \, dx}{x^3 + 2x^2 + 2x + 1} \)

Solution 1: I can't integrate it as it stands - so try to factor the denominator. (Try Synth. Div.)

\[
\begin{array}{c|ccccc}
-1 & 1 & 2 & 2 & 1 \\
\hline
& -1 & -1 & -1 \\
& 1 & 1 & 1 & ( \checkmark )
\end{array}
\]

\[\therefore x+1 \text{ is a factor} \]

and \( x^3 + 2x^2 + 2x + 1 = (x+1)(x^2 + x + 1) \)

Thus, the integrand becomes \( \frac{x}{(x+1)(x^2 + x + 1)} \) which equals, by the work we did in class (*) after

\[ \int \frac{x+1}{x^2 + x + 1} \, dx = \int \frac{1}{x+1} \, dx - \int \frac{1}{x+1} \, dx \]

Thus \( \int \frac{x \, dx}{x^3 + 2x^2 + 2x + 1} = \int \frac{x+1}{x^2 + x + 1} \, dx - \int \frac{1}{x+1} \, dx \)

So you \textbf{SEE} that we have more work to do —
Namely on \[ \int \frac{x+1}{x^2+x+1} \, dx \quad (****) \].

5. There are several ways to handle this. Here is one way —

Since \( d(x^2+x+1) = (2x+1) \, dx \), I want to “make” my numerator (which is \( (x+1) \, dx \)) look like \( (2x+1) \, dx \).

Start with what you’ve got: \( (x+1) \, dx \)

\[
(x+1) \, dx = \frac{1}{2} (x+1) \, dx = \frac{1}{2} (2x+2) \, dx = \frac{1}{2} (2x+1+1) \, dx
\]

\[ = \frac{1}{2} (2x+1) \, dx + \frac{1}{2} \, dx \]

\[ \therefore (****) \text{ becomes:} \]

\[
\int \frac{x+1}{x^2+x+1} \, dx = \frac{1}{2} \int \frac{2x+1}{x^2+x+1} \, dx + \frac{1}{2} \int \frac{dx}{x^2+x+1}
\]

\[ = \frac{1}{2} \int \frac{du}{u} + \frac{1}{2} \int \frac{dx}{(x+1/2)^2 + 3/4}
\]

\[
\text{where } u = \frac{x+1}{\sqrt{3}} \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad dx = \frac{\sqrt{3}}{3} \, du
\]

\[ = \frac{1}{2} \ln \left| x^2 + x + 1 \right| + \frac{1}{2} \frac{\sqrt{3}}{3} \tan^{-1} \left( \frac{2x+1}{\sqrt{3}} \right) + C
\]

6. Thus \[ \int \frac{x \, dx}{x^3+2x^2+2x+1} = \frac{1}{2} \ln \left| x^2 + x + 1 \right| + \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{2x+1}{\sqrt{3}} \right) - \ln |x+1| + C\]