§ 15.5 Chain Rule, p. 967.

A In CALC I we had several different ways to write the chain rule.

One way was:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \quad \text{(Assumption } y = y(x(t))\text{)}$$

B [p. 968] Chain Rule — Case 1

$\begin{align*}
Z &= f(x, y) \\
x &= g(t) \quad [x = x(t)] \\
y &= h(t) \quad [y = y(t)]
\end{align*}$

$Z$ is a differentiable function of $x$ & $y$ and $x$ & $y$ are differentiable functions of $t$;

then

$$\frac{dz}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy} \cdot \frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

What does "differentiability" mean?

1 In "CALC I" A function $y = f(x)$ is differentiable at a point, $a \in \text{dom} f$, iff

$$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h} \text{ exists.}$$

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} \quad \text{ or } \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$
\[ f'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \]

\[ f'(a) \approx \frac{\Delta y}{\Delta x} \quad f'(a) = \frac{\Delta y}{\Delta x} + \varepsilon \]

\[ f'(a) \Delta x = \Delta y + \varepsilon \]

\[ \Delta y = f'(a) \Delta x - \varepsilon \]

\[ f(x) - f(a) = f'(a) \Delta x - \varepsilon \]

\[ f(x) = f(a) + f'(a) \Delta x + \varepsilon \quad \text{ (new } \varepsilon = -\text{ old } \varepsilon) \]

\[ f(x) \approx f(a) + f'(a) \Delta x \]

\[ f(x) \approx f(a) + f'(a)(x-a) \]

Compare w/ \[ f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \]

2. Look at the def. of "differentiable" on p. 962.

If \( z = f(x,y) \), then \( f \) is differentiable at a point \((a,b) \in \text{dom} f \) iff \( \Delta z = f(a+\Delta x, b+\Delta y) - f(a,b) \) can be expressed in the form

\[ \Delta z = f_x(a,b) \Delta x + f_y(a,b) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \]

where \( \varepsilon_1 \to 0 \) as \((\Delta x, \Delta y) \to (0,0)\) and \( \varepsilon_2 \to 0 \) "" "" "" "" "".
Thm 8 p. 962.

If \( z = f(x,y) \) and \( f_x \) exists "near" \((a,b)\) and \( f_y \) exists "near" \((a,b)\) and \( f_x \) is continuous at \((a,b)\) and \( f_y \) is continuous at \((a,b)\),

then \( z = f(x,y) \) is differentiable at \((a,b)\).

Example: \( z = f(x,y) = x^2 + xy - y^2 \)
and \( a = 1, \ b = 2 \) \((a,b) = (1,2)\)

Show that \( f(x,y) \) is differentiable at \((1,2)\).

\[ \begin{align*}
\text{Soln:} & \quad f_x(x,y) = 2x + y \land f_y(x,y) = x - 2y, \\
& \text{both of which exist near and are continuous at} \ (1,2).
\end{align*} \]

Thus by Thm 8 (above) \( z = f(x,y) \) is differentiable at \((1,2)\).

But let's get into this a little more deeply. Look at the definition of differentiability.

1. a) Consider \( f(1+\Delta x, 2+\Delta y) \)

\[ = (1+\Delta x)^2 + (1+\Delta x)(2+\Delta y) - (2+\Delta y)^2 \]
\[ = 1 + 2\Delta x + (\Delta x)^2 + 2 + \Delta y + 2\Delta x + \Delta x \Delta y - 4 - 4\Delta y - (\Delta y)^2 \]

b) \( f(1+\Delta x, 2+\Delta y) - f(1,2) \)

\[ = [4\Delta x + (\Delta x)^2 - 3\Delta y + \Delta x \Delta y - (\Delta y)^2 - 1] - [1 + 2 - 4] \]
\[ = [4 \Delta x - 3 \Delta y + (\Delta x)^2 + \Delta x \Delta y - (\Delta y)^2] \]
\[ = 4 \Delta x - 3 \Delta y + \Delta x \cdot \Delta x + (\Delta x - \Delta y) \Delta y \ (\star) \]
Now look at this:
\[ f_x(x,y) = 2x + y \quad \therefore f_x(1,2) = 2 + 2 = 4 \] and
\[ f_y(x,y) = x - 2y \quad \therefore f_y(1,2) = -3 \]

Let \( E_1 \) and \( E_2 \) be defined as follows:
\[ E_1 = \Delta x \quad \text{and} \quad E_2 = \Delta x - \Delta y \]

Furthermore, in this example
\[ \Delta z = f(1 + \Delta x, 2 + \Delta y) - f(1,2) \]

So.... putting steps \( c, d, e \) together into step \( b \)
\[ \Delta z = f_x(1,2) \Delta x + f_y(1,2) \Delta y + E_1 \Delta x + E_2 \Delta y \]

and furthermore, since \( E_1 = \Delta x \) and \( E_2 = \Delta x - \Delta y \),
as \( (\Delta x, \Delta y) \to (0,0) \) it follows that both \( E_1 \to 0 \)
and \( E_2 \to 0 \).

Thus by the very definition of differentiability,
it follows that \( z = f(x,y) = x^2 + xy - y^2 \) is differentiable at \( (1,2) \).