1. Problems — Requested by Students/Teacher.

A. S15.7: p.998: #37: Find the shortest distance from the point \(P(2,1,-1)\) to the plane \(\Pi: x+y-z=1\).

Solution

1. \(d(P,\Pi) = \sqrt{(x-2)^2 + (y-1)^2 + (z+1)^2}\)

\[ z = x+y-1 \]

\[ d(P,\Pi) = \sqrt{(x-2)^2 + (y-1)^2 + (x+y)^2} \]

\[ d^2(P,\Pi) = (x-2)^2 + (y-1)^2 + (x+y)^2 \]

Let \(f(x,y) = (x-2)^2 + (y-1)^2 + (x+y)^2 \) (*)

2. Minimize \(f(x,y)\).

\[ f_x = 2(x-2) + 2(x+y) = 2x - 4 + 2x + 2y = 4x + 2y - 4 \]

\[ f_{xx} = 4 \quad f_{xy} = 2 \]

\[ f_y = 2(y-1) + 2(x+y) = 2y - 2 + 2x + 2y = 2x + 4y - 2 \]

\[ f_{yy} = 4 \]

\[ D = f_{xx}(x,y) f_{yy}(x,y) - [f_{xy}(x,y)]^2 = 4 \cdot 4 - (2)^2 = 16 - 4 = 12 > 0 \]

3. \[ 0 = f_x(x,y) = 4x + 2y - 4 \]

\[ -2 \times 0: -8x - 4y + 8 = 0 \]

\[ 0 = f_y(x,y) = 2x + 4y - 2 \]

\[ 2 \times 0: \frac{2x + 4y - 2}{-6x + 6 = 0} \]

\[ x = 1 \]

\[ 4x + 2y - 4 = 0 \]

\[ x = -1 \]

\[ 2y \]

\[ y = 0 \]

\[ z = x + y - 1 = 1 + 0 - 1 = 0 \]

CP: \((1,0)\)
\[ f_{xx}(1,0) = 4 > 0 \text{ and } D(1,0) = 12 > 0 \]

so by the 2nd Der. Test \( f(x,y) \) has a loc. min value at \((1,0)\).

3. So the point \( Q(1,0,0) \) ETI and \( d(P,Q) \) is minimal.

4. \[
d(P,Q) = \sqrt{(2-1)^2 + (1-0)^2 + (-1-0)^2} = \sqrt{1+1+1} = \sqrt{3} \text{ units.}
\]

B. The shortest distance from \( P(2,1,-1) \) to \( \Pi : x+y-z = 1 \) is \( \sqrt{3} \) units.

**3.** § 15.7; p. 998; #29: Find Abs Max/Min Val. of:

\[
z = f(x,y) = x^2 + y^2 + x^2y + 4
\]

on \( D = \{(x,y) \mid |x| \leq 1, |y| \leq 1 \} \)

**SOLN:** 1. FIG.

2. "Inside" \( D.\)

a. \[ f_x(x,y) = (2x+2xy) \]

b. \[ f_{xx}(x,y) = (2+2y) \]

c. \[ f_{xy}(x,y) = 2x \]

d. \[ f_y(x,y) = (2y+x^2) \]

e. \[ f_{yy}(x,y) = 2 \]

\[ 0 = f_x(x,y) = 2x + 2xy \quad \rightarrow \quad 2x + 2x\left(-\frac{x^2}{2}\right) = 0 \]

\[ 0 = f_y(x,y) = 2y + x^2 \quad \rightarrow \quad y = -\frac{x^2}{2} \quad 2x - x^3 = 0 \]

\[ x(2-x^2) = 0 \]

\[ x = 0, x = \pm \sqrt{2} \]

\[ y = -\frac{x^2}{2} \]

\[ x = \sqrt{2}, y = -1 \]
Critical Points \((0,0), (\sqrt{2},-1), (-\sqrt{2},-1)\)

\[
D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - [f_{xy}(x,y)]^2
\]

\[
= (2+2y)(2) - (2x)^2
\]

\[
= 4 + 4y - 4x^2
\]

<table>
<thead>
<tr>
<th>Point</th>
<th>(D(0,0) = 4 &gt; 0)</th>
<th>(f_{xx}(0,0) = 2 &gt; 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0,0))</td>
<td>Waste Of Time! You Don't need to do 2nd Der Test on Abs Max/Min Problems</td>
<td></td>
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</tbody>
</table>

\[
Z = f(x,y) = x^2 + y^2 + x^2y + 4
\]

\[
f(0,0) = 4
\]

\[
f(\sqrt{2},-1) = 2 + 1 - 2 + 4 = 5
\]

\[
f(-\sqrt{2},-1) = 2 + 1 - 2 + 4 = 5
\]

3. On the boundary of \(D\):

- \(i\) \(x=1\) \(-1 \leq y \leq 1\) \(\text{right bdry}\)

\[
f(1, y) = 1 + y^2 + y + 4 = y^2 + y + 5
\]

\[
\frac{df}{dy} = 2y + 1 = 0 \quad y = -\frac{1}{2} \quad (1,-\frac{1}{2})
\]

\[
f(1, -\frac{1}{2}) = 1 + \frac{1}{4} - \frac{1}{2} + 4 = \frac{3}{4} + \frac{16}{4} = \frac{19}{4}
\]

- \(ii\) \(x = -1\) \(-1 \leq y \leq 1\) \(\text{left bdry}\)

\[
f(-1, y) = 1 + y^2 + y + 4 = y^2 + y + 5
\]

\[
\text{C.m. } y = -\frac{1}{2}
\]

\[
f(-1, -\frac{1}{2}) = 1 + \frac{1}{4} - \frac{1}{2} + 4 = \frac{19}{4}
\]

- \(iii\) \(y = 1\) \(-1 \leq x \leq 1\) \(\text{top bdry}\)

\[
f(x,1) = x^2 + 1 + x^2 + 4 = 2x^2 + 4
\]

\[
f_{xx}(x,1) = 2x \quad x = 0 \quad (x, y) = (0, 1)
\[ f(0,1) = 0 + 1 + 0 + 4 = 5 \]

\[ f(0,-1) = 0 + 1 + 0 + 4 = 5 \]

\[ f(x,-1) = x^2 + 1 - x^2 + 4 = 5 \]

\[ \frac{df(x,-1)}{dx} = 0 \text{ for } -1 \leq x \leq 1 \]

The value is 5 for every \( x \), \(-1 \leq x \leq 1\)

4. Look @ \( f(x,y) \) @ (1,1), (1,-1), (-1,1), (-1,-1)

\[ f(1,1) = 1 + 1 + 1 + 4 = 7 \]

\[ f(1,-1) = 1 + 1 - 1 + 4 = 5 \]

\[ f(-1,1) = 1 + 1 + 1 + 4 = 7 \]

\[ f(-1,-1) = 1 + 1 - 1 + 4 = 5 \]

4. Look @ \( f(x,y) \) @ (1,1), (1,-1), (-1,1), (-1,-1)

The function has an abs. max value of 7 @ (1,1) and (-1,1).

The fun. has an abs. min, value of 4 @ (0,0).