
Recall \( \{ f_i \}_{i=1}^n \) is a LD set \( \iff \exists e_i, i=1,\ldots,n \) 
\[ e_i \text{ at least one } c_i \neq 0 \text{ and } \sum_{i=1}^n c_i f_i(x) = 0 \quad \forall \ x \in I. \]

The implication is that \( \{ f_i \}_{i=1}^n \) is LD on \( I \) \( \iff \) one of the functions can be written as a linear combination of the rest of them.

Example: \( \{ 1, x, x^2, 2x^2 + 3x + 5 \} \) is a LD set on \( I = (-\infty, \infty) \).

Why? B/c
\[ 2x^2 + 3x + 5 = 5(1) + 3(x) + 2(x^2) \]

2. A set \( \{ f_i \}_{i=1}^n \) on \( I \) is LI (linearly independent) on an interval \( I \) iff it is NOT LD on \( I \).

Meaning (loosely) \( \iff \) The only way for
\[ \sum_{i=1}^n c_i f_i(x) = 0 \quad \forall \ x \in I \]
to happen is for \( c_i = 0 \quad \forall \ i \).

Examples: \( \{ e^x, x \} \) is a LI set on \( I = \mathbb{R} \).

Proof: 1. Form linear combo.
\[ c_1 e^x + c_2 x = 0 \quad (\ast) \]
2. Take the diff (=derivative) of b.s.
\[ c_1 e^x + c_2 = 0 \]
3. Again: \( c_1 e^x = 0 \Rightarrow c_1 = 0 \quad \& \quad c_2 = 0 \)
4. Given (\ast) it follows that \( c_1 \) and \( c_2 \) MUST be zero!
Another Example — Another Technique.

Given \( \{ e^x, x \} \).

Reg'd: Dep or Indep? on \((-\infty, \infty)\)

Solution

1. Form LinComb: \( c_1 e^x + c_2 x = 0 \) (*)

2. Subs \( x = 0 \): \( c_1 e^0 + c_2(0) = 0 \Rightarrow c_1 = 0 \)
   \(\therefore\) \((*)\) becomes \( c_2 x = 0 \)
   Now subs \( x = 1 \): \( c_2 = 0 \)

3. Yada, yada, yada... \( \{ e^x, x \} \) is LinIndep.

Review of "Necessary" and "Sufficient" Conditions.

A. If \( H \), then \( C \): \( H \Rightarrow C \) (Conditional Statement)

   \(\uparrow\)    \(\uparrow\)
   Suff.    Necess

B. Converse of Conditional: \( C \Rightarrow H \)

C. Note: The validity of any given Conditional Statement does NOT guarantee the validity of its converse.

VI. Wronskian. — The following Thm provides a Sufficient condition for the Linear Independence.
or a set of \( n \) functions \( \{ f_i \}_{i=1,...,n} \) on \( I \).

(Each func. is assumed to be differentiable at least \( n-1 \) times.)

Define

\[
W( f_1, f_2, ..., f_n ) = \begin{vmatrix}
  f_1 & f_2 & \ldots & f_n \\
  f_1' & f_2' & \ldots & f_n' \\
  \vdots & \vdots & \ddots & \vdots \\
  f_1^{(n-1)} & f_2^{(n-1)} & \ldots & f_n^{(n-1)}
\end{vmatrix}
\]

\[ n \times n \]

to be continued,...