No class Fri.

8.4.1.3 — SOLUTIONS TO LINEAR EQUATIONS (p. 120)

A. General Form (Standard Form)

\[
\sum_{i=0}^{n} a_{n-i}(x) \frac{d^{(n-i)}}{dx^{(n-i)}} y(x) = 0 \quad (\ast) \quad \text{(Homogeneous)}
\]

\[
\sum_{i=0}^{n} a_{n-i}(x) \frac{d^{(n-i)}}{dx^{(n-i)}} y(x) = g(x) \quad (\ast\ast) \quad \text{(Non-Hom.)}
\]

B. Simplifying Assumptions (for rest of text)

- \(a_2(x)\) are continuous
- \(g(x)\) is continuous
- \(a_n(x) \neq 0 \quad \forall x \in I.\)


(Example: If \(y_1, y_2, y_3\) are solutions of

\[a_5 y^{(5)} + a_4 y^{(4)} + a_3 y^{(3)} + a_2 y'' + a_1 y' + a_0 y = 0, \quad (\ast\ast)\]

Then \(y = c_1 y_1 + c_2 y_2 + c_3 y_3\) is also a solution of \((\ast\ast\ast)\)

D. Ex 18 (p. 121) \(\phi_1(x) = x^2\) \(\frac{d}{dx}\) \(\phi_2(x) = x^2 \ln(x)\)

are solutions to \(x^3 y''' - 2x y' + 4y = 0\) \((\ast)\).

\(\therefore \phi = c_1 \phi_1 + c_2 \phi_2\) is also a solution to \((\ast)\).

Sol. (Verification)
\[\text{Check: } \phi_1(x) = x^2; \quad \phi_1'(x) = 2x, \quad \phi_1''(x) = 2\]
and \(\phi_1'''(x) = 0\). Substituting into (\star)
\[x^3 \phi_1''' - 2x \phi_1' + 4 \phi_1 = 0 - 2x(2x) + 4x^2 = 0\]

Check \(\phi_2(x) = x^2 \ln(x); \quad \phi_2'(x) = 2x \ln(x) + x^2 \frac{1}{x}\)
\[\phi_2'(x) = 2x \ln(x) + x\]
\[\phi_2''(x) = 2 \left( \ln(x) + \frac{x}{x} \right) + 1 = 2 \ln(x) + 1 + 1 = 2 \ln(x) + 3\]
\[\phi_2'''(x) = \frac{2}{x}\]

\[\text{Subs: } x^3 \phi_2''' - 2x \phi_2' + 4 \phi_2 = 0\]
\[x^3 \left( \frac{2}{x} \right) - 2x \left( 2x \ln(x) + x \right) + 4x^2 \ln(x) = 0\]
\[= 2x^2 - 4x^2 \ln(x) - 2x^2 + 4x^2 \ln(x) = 0\]

2. Now do the example: \(\phi = c_1 \phi_1 + c_2 \phi_2\) is also a sol.
Show by subs:
\[x^3 \phi''' - 2x \phi' + 4 \phi = x^3 (c_1 \phi_1''' + c_2 \phi_2''')\]
\[- 2x \left( c_1 \phi_1' + c_2 \phi_2' \right) + 4 \left( c_1 \phi_1 + c_2 \phi_2 \right)\]
\[= c_1 \left( x^3 \phi_1''' - 2x \phi_1' + 4 \phi_1 \right) + c_2 \left( x^3 \phi_2''' - 2x \phi_2' + \phi_2 \right)\]
\[= c_1 0 + c_2 0 = 0\]
\[\therefore \phi(x) = c_1 \phi_1(x) + c_2 \phi_2(x) \text{ is a sol. to (\star).}\]
Linearly Independent Solutions. (P. 122).

\[ \{ y_1, y_2, \ldots, y_n \} \text{ is a set of solutions to an } n\text{-th order HLODE} \]

Then \( \{ y_1, y_2, \ldots, y_n \} \text{ is LIN. INDEP.} \)

\[ \iff (\quad) \quad (\text{necessary and sufficient}) \]

\[ W(y_1, y_2, \ldots, y_n)(x) \neq 0 \quad \forall x \in I \]

Fundamental Set of Solutions to a HLODE. (P123)

A set \( \{ y_1, \ldots, y_n \} \) of solutions to \( n\)-th order HLODE on \( I \) is FUND. iff it is LIN. Indep. (i.e. \( W(y_1, \ldots, y_n)(x) \neq 0 \quad \forall x \in I \)).

i.e. All solutions "look like" this:

\[ y = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x). \]