Consider \( y' - y = 0 \). Solve this:

**Solution**

1. If, \( \mu(x) = e^{\int -1 \, dx} = e^{-x} \)

2. Apply & reduce

\[
\begin{align*}
  e^{-x}(y' - y) &= e^{-x}(0) \\
  (e^{-x}y)' &= 0 \\
  e^{-x}y &= C \\
  y &= Ce^x
\end{align*}
\]

Also

\[
\begin{align*}
  y' - y &= 0 \\
  \text{Aux Eq: } m - 1 &= 0 & \Rightarrow & & \{1\} \\
  \therefore \text{ Gen Sol } & y = c_1 e^x
\end{align*}
\]

Why bother? Because \( y = c_1 e^x \) is known to be a sol. to \( y' - y = 0 \)

So consider the related differential operator

\[
L(y) = (D - 1)(y)
\]

So

\[
\begin{align*}
  L(c_1 e^x) &= (D - 1)(c_1 e^x) = D(c_1 e^x) - c_1 e^x \\
  &= c_1 e^x - c_1 e^x = 0
\end{align*}
\]
\[ L(y) = (D-1)(y) \text{ is an Annihilator of } ce^x \]

II. \[ \text{Consider } y = xe^x \]

Humm... \[ y'' - 2y' + y = 0 \] \text{(Type II)}

Solve:
1. \[ \text{AuxEq: } m^2 - 2m + 1 = 0 \]
2. \[ \text{Solve: } (m-1)^2 \]
3. \[ \text{Gen Sol: } y = c_1 e^x + c_2 xe^x \]

This means \[ L(y) = (D^2 - 2D + 1)(y) \]
Annihilates BOTH \( e^x \) and \( xe^x \)!

B. So we can generalize and say that
\[ (D- \alpha)^n \text{ Annihilates } e^{\alpha x}, xe^{\alpha x}, \ldots, x^{n-1}e^{\alpha x} \]

III. \[ \text{How do we use this knowledge to solve LODE's which are NOT homogeneous?} \]

B. Example: Solve \[ y' + y = x \] using Annihilator Method (AM)

Solve: Find an Annihilator of the forcing function
i.e. Annihilate “\( x \)” \[ L(x) = 0 \] What is \( L \)?
Let \( L(y) = D^2(y) \). Then \( L(x) = D^2(x) = 0 \)

\[ D^2 \text{ annihilates } x. \]

2. Write the LODE in Op. Form: \( (D+1)(y) = x \) \((\star)\)

3. (Here's Where I goofed in class!)
   Solve the related homogeneous problem:
   \( (D+1)(y) = 0 \) \((\star\star)\)
   And we get the complementary solution, which is called \( y_c \)

   So \[ y_c = c_1 e^{-x} \]

4. Apply Annihilator to b.s. of \((\star)\)
   \[ D^2(D+1)(y) = D^2(x) = 0 \]

5. Solve the homogeneous eq. \[ D^2(D+1)(y) = 0 \] \((\star\star)\)
   a. AuxEq. \( m^2(m+1) = 0 \); roots \(-1, 0 \text{ (mult. 2)}\)
   b. Sol. \[ y = c_1 + c_2 x + c_3 e^{-x} \]

   But this term is just the complementary solution to \((\star)\).

5. Thus a particular solution to \((\star)\) should have the form
   \[ y_p = A + Bx \]
   (notice we use the subscript "p" and change the \( c_1 \) and \( c_2 \) to \( A \) and \( B \)).
   These are the "Undetermined Coefficients"
Since $y_p$ should be a sol to (*),

$$(D+1)(y_p) = x$$

$\Rightarrow \quad y_p' + y_p = x \quad \Rightarrow \quad (A+Bx)' + (A+Bx) = x$

$\Rightarrow \quad B + A + Bx = x$

Look at this equation — $2$ equal polynomials, so $B + A = 0$ and $B = 1$ — so $A = -1$

$\Rightarrow \quad$ The exact nature of $y_p$ is established, and

\[ y_p = -1 + x \]

$\therefore$ Thus the gen sol to the original equation $y' + y = x$

\[ y = y_c + y_p = c_1e^{-x} + x - 1 \]

Check. $\quad y = c_1e^{-x} + x - 1$,

$y' = -c_1e^{-x} + 1$

and

$y' + y = (-c_1e^{-x} + 1) + (c_1e^{-x} + x - 1) = x \quad \checkmark$

Have a great Spring Break.