A Bit More of G.S. (§12.3: Ex8, p.802)

Repeating decimal representations.

1. \(0.9\overline{9}\) ← write this as a rational number.

**Tech. Sol.**

\[0.9 = \frac{9}{10}\]
\[0.99 = \frac{99}{100} = \frac{9}{100} + \frac{90}{100} = \frac{90}{100} + \frac{9}{100}\]
\[= \frac{9}{10} + \frac{9}{100}\]

\[0.999 = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000}\]
\[= \frac{9}{10} + \frac{1}{10} \cdot \frac{9}{100} + \frac{1}{100} \cdot \frac{9}{10}\]

Well, look at \(\frac{\frac{9}{1000}}{\frac{9}{100}} = \frac{\frac{9}{100}}{9} = \frac{1}{10}\) \(\Rightarrow r\)

So what we got is [A] G.S. \(\frac{9}{10}, \frac{9}{100}, \frac{9}{1000}, \ldots\)
and a geom series (infinite)

\[a_n = a_1 r^{n-1}\]

So here \(a_n = \frac{9}{10} \left(\frac{1}{10}\right)^{n-1}\)

\[S_\infty = \frac{a}{1 - r} = \frac{9/10}{1 - \frac{1}{10}} = \frac{\frac{9}{10}}{\frac{9}{10}} = 1\]

\[\therefore 0.9\overline{9} = 1\]
2. Convert (using short cut) to a rational number

\[ 0.123123123 \]

**Solution**

1. Let \( x = 0.123123123 \)

2. \( 1000x = 123.123123 \)

3. **Diff.**

   \[
   1000x = 123.123123 \\
   - x = -0.123123 \\
   999x = 123 \\
   \therefore \quad x = \frac{123}{999}
   \]

3. Convert \( 45.1010 \)

**Solution**

1. Let \( x = 45.1010 \)

2. \( 100x = 451.01010 \)

3. **Sub** \( 99x = 446.5 \)

   \[
   \therefore \quad x = \frac{446.5}{99}
   \]

3. §12.4: PMI (Principle of Mathematical Induction)

   (p. 809)

**A** To prove a theorem by PMI, 3 steps are required.

1. **prove equation is true for** \( n=1 \)

2. **prove that** \( [ \text{If the eq. is true for } n=k, \text{then the eq. is true for } n=k+1 ] \)

   (Step 2 is a proof within a proof).
The conclusion that the eq. is true for all natural numbers, \( n \).

\[ P(n) \]

**B** Prove, using PMI, \( 1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2} \)

**Proof**

1. Verify the formula for \( n = 1 \)

\[ 1 = \frac{1 \cdot 2}{2} = 1 \checkmark \]

2. Assume \( P(k) \) is true. Prove \( P(k+1) \) is true.

\[ 1 + 2 + 3 + \ldots + k + (k+1) = \frac{k(k+1)}{2} \]

Prove: \( 1 + 2 + 3 + \ldots + (k+1) = \frac{(k+1)(k+2)}{2} \)

\[ \text{Proof within Proof.} \]

\[ 1 + 2 + 3 + \ldots + (k+1) = 1 + 2 + 3 + \ldots + k + (k+1) \]

\[ = (1 + 2 + 3 + \ldots + k) + (k+1) \]

\[ = \frac{k(k+1)}{2} + (k+1) \]

\[ \text{Subs} \]

\[ = \frac{k(k+1) + 2(k+1)}{2} \]

\[ = \frac{(k+1)(k+2)}{2} \]

\[ \text{Inductive Hypothesis} \]

Therefore \( 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \) is true for all natural numbers \( n \).
3. \[ 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \quad \forall n \in \mathbb{Z}^+ \]

where \( \forall = "\text{for all}" \quad \mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\} \)

\[ \square \] §12.4; p. 812: #5: Prove Using PMI

\[ 2 + 5 + 8 + \ldots + (3n-1) = \frac{1}{2} n (3n+1) \]

Proof

1. \( n = 1 \):

\[ 2 = \frac{1}{2} (1)(3(1)+1) = \frac{1}{2} \cdot 1 \cdot 4 = 2 \checkmark \]

P(1)

2. Pf w/o Pf.

Assume \( P(k) \):

\[ 2 + 5 + 8 + \ldots + (3k-1) = \frac{1}{2} k(3k+1) \]

Prove \( P(k+1) \):

\[ 2 + 5 + 8 + \ldots + (3(k+1)-1) = \frac{1}{2} (k+1)(3(k+1)+1) \]

\[ = \frac{1}{2} (k+1)(3k+4) \]

Proof

\[ 2 + 5 + 8 + \ldots + (3(k+1)-1) \]

\[ = \left[ 2 + 5 + 8 + \ldots + (3k-1) \right] + (3(k+1)-1) \]

\[ = \left[ \frac{1}{2} k(3k+1) \right] + (3(k+1)-1) \]

Subs

\[ \frac{k(3k+1)}{2} + \frac{2(3(k+1)-1)}{2} \]

\[ = \frac{3k^2 + k + 2(3k+3-1)}{2} = \frac{3k^2 + 7k + 4}{2} \]

\[ = \frac{3k^2 + 7k + 4}{2} = \frac{(k+1)(3k+4)}{2} = \frac{1}{2} (k+1)(3(k+1)+1) \]

\[ \square \]

3. \[ 2 + 5 + \ldots + (3n-1) = \frac{1}{2} n (3n+1) \quad \forall n \in \mathbb{Z}^+ \]