II §14.4: Using the C.R. to derive a formula for I.D.

(Implicit Differentiation)

[A] \( y = f(x) \) \hspace{1cm} (Graph is in 2-Space)

Define \( F(x, y) : W = F(x, y) = F(x, f(x)) \)

\[ = F(x, y(x)) \]

and the original function, \( y = f(x) \), is simply the level curve of \( W = F(x, y) \) @ \( W = 0 \).

[B] Consider \[ \frac{\partial W}{\partial x} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dx} \hspace{1cm} (\ast) \]

But the Level Curve Eq is

\[ W = F(x, y) = 0 \]

\[ \therefore \frac{\partial w}{\partial x} = 0 \hspace{1cm} (\ast\ast) \]

\[ \therefore \frac{\partial w}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dx} = 0 \hspace{1cm} (\ast\ast\ast) \]

But \( \frac{dx}{dx} = 1 \) and by way of notation \( \frac{\partial w}{\partial x} = F_x \)

and \( \frac{\partial w}{\partial y} = F_y \)

\[ \therefore (\ast\ast\ast) \hspace{1cm} becomes \hspace{1cm} F_x + F_y \frac{dy}{dx} = 0 \hspace{1cm} (\ast) \]

[C] Solving \( (\ast) \) for \( \frac{dy}{dx} \):

\[ \frac{dy}{dx} = -\frac{F_x}{F_y} \]
Example.

\[ x^2 + y^2 = 25 \] (*)

Find Eq. of tangent line when \( x = \frac{1}{3} \).

**Solution using 8.14.4 method.**

1. \( F(x, y) = x^2 + y^2 - 25 \). Then (*) is just \( F(x, y) = 0 \).

2. \( F_x = 2x, \quad F_y = 2y \)

3. \( \frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y} \)

4. \( \frac{dy}{dx} \bigg|_{x=\frac{1}{3}} = -\frac{1}{4\sqrt{14}} \)

\[ \left(\frac{1}{3}\right)^2 + y^2 = 25 \]
\[ y^2 = 25 - \frac{1}{9} \]
\[ y = \sqrt{\frac{224}{9}} \]
\[ = \frac{2}{3} \sqrt{56} \]
\[ = \frac{4}{3} \sqrt{14} \]

**P(\frac{1}{3}, \frac{4\sqrt{14}}{3})**

5. \( y - y_1 = m(x - x_1) \)

\[ y - \frac{4\sqrt{14}}{3} = -\frac{1}{4\sqrt{14}} \left( x - \frac{1}{3} \right) \]

\[ y = -\frac{1}{4\sqrt{14}} x + \frac{1}{12\sqrt{14}} + \frac{4\sqrt{14}}{3} \]

\[ y = \left( -\frac{1}{4\sqrt{14}} \right) x + \frac{225\sqrt{14}}{168} \]

or \[ y = -\frac{\sqrt{14}}{56} x + \frac{225\sqrt{14}}{168} \]

\[ \frac{1}{12\sqrt{14}} + \frac{4\sqrt{14}}{3} \]
\[ = \frac{\sqrt{14}}{168} + \frac{224\sqrt{14}}{168} \]

Finished After Class.
Expand this Idea to 3-Space.

\[ Z = f(x, y) \quad \text{in 4-space} \quad \Rightarrow \quad W = F(x, y, z(x, y)) \]

Now the surface in 3-space is simply \( F(x, y, z) = 0 \)

\( \frac{\partial W}{\partial x} = \frac{\partial W}{\partial z} \cdot \frac{\partial z}{\partial x} + \frac{\partial W}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial W}{\partial z} \cdot \frac{\partial z}{\partial x} \)

\( \frac{\partial W}{\partial z} = \frac{\partial W}{\partial x} \cdot 1 + \frac{\partial W}{\partial y} \cdot 0 + \frac{\partial W}{\partial z} \cdot \frac{\partial z}{\partial x} \)

\( = F_x + F_z \frac{\partial z}{\partial x} \)

\( \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \)

\( \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \)

Example: §14.4: p.988: #30: (part i) Find \( \frac{\partial z}{\partial x} \)

if \( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0 \) @ \((2,3,6)\)

Solution:

1. \( F(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 \)

2. \( F_x = -\frac{1}{x^2} \quad \text{and} \quad F_z = \frac{1}{z^2} \)

3. \( \left. \frac{\partial z}{\partial x} \right|_{(2,3,6)} = -\frac{F_x}{F_z} \right|_{(2,3,6)} = -\frac{\frac{1}{2^2}}{\frac{1}{6^2}} \right|_{(2,3,6)} = -\frac{36}{4} = -9 \)