In §14.7: (p. 1012)

A. Let \( f(x,y) \) be defined on a region \( R \) containing the point \((a,b)\). Then

1. \( f(a,b) \) is a **local maximum value** of \( f \) if
   
   \[ f(a,b) \geq f(x,y) \text{ for all domain points } (x,y) \text{ in an open disk centered at } (a,b). \]

2. \( f(a,b) \) is a **local minimum value** of \( f \) if
   
   \[ f(a,b) \leq f(x,y) \text{ for all domain points } (x,y) \text{ in an open disk centered at } (a,b). \]

B. **Open Disk** of radius \( \epsilon > 0 \), centered @ \((a,b)\)

\[ D(a,b,\epsilon) = \{(x,y) \mid (x-a)^2 + (y-b)^2 < \epsilon^2 \} \]

C. **Thm 10 (p. 1012) [First Derivative Test]**

**IF** \( \exists = f(x,y) \) has a loc max/min @ \((a,b) \in \text{Int}(\text{Domain}) \)

and **IF** \( z_x \land z_y \) exist there, \( (z_x = f_x, z_y = f_y) \)

**THEN** \( f_x(a,b) = 0 = f_y(a,b) \)

D. **Critical Points** (p. 1013). -- \( \exists = f(x,y) \) and \( D = \text{dom}(f) \) and \( (x,y) \in \text{Int}(D) \) s.t.

(i) \( f_x(x,y) = 0 \) and \( f_y(x,y) = 0 \).

OR

(ii) either \( f_x(x,y) \) DNE or \( f_y(x,y) \) DNE

Then \((x,y)\) is called a **critical point** (cp.).
### Saddle point (Paraphrasing)

A function \( z = f(x, y) \) has a saddle point at \((a, b)\)

if there are points \((x, y)\) "near" \((a, b)\) s.t.

at some \((x, y)\) \( f(x, y) > f(a, b) \) and at other

points \((x, y)\) \( f(x, y) < f(a, b) \)

### The 2nd Der. Test. (P. 1014)

1. **Def.**: "Discriminant", "Hessian"

   \[ z = f(x, y) \] \( \text{If all 1st \& 2nd partials are cont.} \)

   \[ \Delta = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} \cdot f_{yy} - f_{xy} \cdot f_{yy} \]

   \[ = f_{xx} \cdot f_{yy} - [f_{xy}]^2 \]

   and \( \Delta(a, b) = f_{xx}(a, b) \cdot f_{yy}(a, b) - [f_{xy}(a, b)]^2 \)

2. **IF**
   
   1. \( z = f(x, y) \) and all first \& 2nd partial derivs.
      are cont. on some open disk centered @
      \((a, b)\) and

      2. \( f_x(a, b) = 0 = f_y(a, b) \),

   **THEN**

   \[ \Delta(a, b) > 0 \begin{cases} \frac{1}{2} \text{ f has a loc. max. val. @ (a, b) } & \text{IF} \\ f_{xx}(a, b) < 0 \text{ and } \Delta(a, b) > 0. \end{cases} \]

   \[ \Delta(a, b) < 0 \begin{cases} 2. \text{ f has a loc. min. val. @ (a, b) } & \text{IF} \\ f_{xx}(a, b) > 0 \text{ and } \Delta(a, b) > 0. \end{cases} \]

   \[ \Delta(a, b) = 0 \begin{cases} 3. \text{ f has a saddle point } & \text{IF} \\ \Delta(a, b) < 0. \end{cases} \]

   \[ \Delta(a, b) = 0 \begin{cases} 4. \text{ The Test is inconclusive. } & \text{IF} \\ \Delta(a, b) = 0. \end{cases} \]
Examples:


\[ z = f(x, y) = 2x^2 + 3xy + 4y^2 - 5x + 2y \]

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CLASS ENDS — AFTER CLASS...

Solution:

1. Find "First Partials."

\[ f_x(x, y) = 4x + 3y - 5; \quad f_y(x, y) = 3x + 8y + 2 \]

We'll need to use these several times.

2. Set "1st Partials" equal to zero and solve simultaneously:

\[
\begin{align*}
4x + 3y - 5 &= 0 \\
3x + 8y + 2 &= 0
\end{align*}
\]

\[
\begin{align*}
4x + 3y &= 5 \\
3x + 8y &= -2
\end{align*}
\]

I solve such things using a "mental Cramer's Rule." — Remind me to fill you in on the details on Mon.

\[ x = \frac{46}{23} \quad y = -1 \]

\[ P(2, -1) \]

\[ f_x(2, -1) = 0 = f_y(2, -1) \]

3. Next find \( \Delta (z, -1) \):

\[ f_{xx}(x, y) = 4; \quad f_{yy}(x, y) = 8; \quad f_{xy}(x, y) = 3. \]

\[ f_{xx}(2, -1) = 4; \quad f_{yy}(2, -1) = 8 \quad \text{and} \quad f_{xy}(2, -1) = 3. \]

\[ \Delta (2, -1) = f_{xx}(2, -1) f_{yy}(2, -1) - [f_{xy}(2, -1)]^2 = 4 \cdot 8 - 3^2 = 32 - 9 > 0. \]

The Exact Value is irrelevant to this investigation.

4. \[ f_{xx}(2, -1) = 4 > 0 \]

5. Thus by the 2nd Der. Test, \( f(x, y) \) has a loc. min. val. \( @ P(2, -1) \), and

\[ f(2, -1) = 2(2)^2 + 3(2)(-1) + 4(-1)^2 - 5(2) + 2(-1) = 8 - 6 + 4 - 10 - 2 = -12 - 18 = -30 \]

6. \( f(x, y) = 2x^2 + 3xy + 4y^2 - 5x + 2y \) has a loc. min. val. of -6 at \( P(2, -1) \).