We looked at the write-up of the material in the first few pages of 515.1 pp. 1051-53.

There is new MML homework.

Notation in Prep. for "Fubini."

\[ \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx \]

\[ \text{def} \quad \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x, y) \, dy \, dx \]

\[ \text{def means} \quad \int_{x=a}^{x=b} \left[ \int_{y=c}^{y=d} f(x, y) \, dy \right] \, dx \]

Example: Evaluate.

\[ \int_{x=1}^{x=5} \int_{y=1}^{y=3} xy \, dy \, dx \]

Solution:

\[ \int_{x=1}^{x=5} \int_{y=1}^{y=3} xy \, dy \, dx = \int_{x=1}^{x=5} \left[ \int_{y=1}^{y=3} xy \, dy \right] \, dx \]

\[ = \int_{x=1}^{x=5} \left\{ \frac{xy^2}{2} \right\}_{y=1}^{y=3} \, dx = \int_{x=1}^{x=5} \left( \frac{9}{2} - \frac{1}{2} \right) \, dx \]

\[ = 4 \int_{x=1}^{x=5} x \, dx = 4 \left[ \frac{x^2}{2} \right]_{x=1}^{x=5} = 48 \]
II. **Fubini's Theorem (First Form) "Baby Fubini" (p. 1054)**

*IF* $f(x,y)$ is continuous on $R = [a,b] \times [c,d]$, *then*

\[
\int \int_R f(x,y) \, dA = \int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y) \, dx \, dy
\]

also

\[
= \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x,y) \, dy \, dx
\]

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2. **ADYR (Always Draw Your Region).**

![Diagram of a rectangular region](image)

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3. **§ 15.1: p. 1063: #1**

\[
\int_{x=0}^{x=3} \int_{y=0}^{y=2} (4-y^2) \, dy \, dx
\]

*So [hint]*

\[
\int_{x=0}^{x=3} \int_{y=0}^{y=2} (4-y^2) \, dy \, dx = \int_{x=0}^{x=3} \left\{ 4y - \frac{y^3}{3} \right\}_{y=0}^{y=2} \, dx
\]

\[
= \int_{x=0}^{x=3} \frac{16}{3} \, dx = \frac{16}{3} \int_{x=0}^{x=3} 1 \, dx = \frac{16}{3} \checkmark
\]

PS: What did I *FAIL* to do in the problem above? I didn't draw the region. 😐
III. Preview of Stronger Fubini.

515.1: P. 1063: #13. Integrate \( f(x, y) = x^2 + y^2 \) over the triangle with vertices \((0, 0), (1, 0), (0, 1)\).

Solve.

1. \( \int \int_{R} x^2 + y^2 \, dA \)

2. \( R = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} x^2 + y^2 \, dy \, dx \)

As \( x \) "moves" from 0 to 1, \( y \) "moves" from 0 to \( 1 - y \).

\[ \int_{x=0}^{x=1} \left( \int_{y=0}^{y=1-x} x^2 y + \frac{y^3}{3} \bigg|_{y=0}^{y=1-x} \right) \, dx \]

\[ = \int_{x=0}^{x=1} \left( x^3 (1-x) + \frac{(1-x)^3}{3} \right) \, dx \]

Recall:\n\[ (A-B)^3 = A^3 - 3A^2B + 3AB^2 - B^3 \]

\[ = \int_{x=0}^{x=1} \left[ x^2 - x^3 + \frac{1}{3} \left( 1 - 3x + 3x^2 - x^3 \right) \right] \, dx \]

\[ = \int_{x=0}^{x=1} \left( \frac{1}{3} - x + 2x^2 - \frac{4}{3} x^3 \right) \, dx \]

"Easy" subs.

\[ = \left[ \frac{1}{3}x - \frac{x^2}{2} + \frac{2}{3} x^3 - \frac{1}{3} x^4 \right]_{x=0}^{x=1} = \frac{1}{3} - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} = \frac{2}{3} - \frac{1}{2} = \frac{1}{6} \]

Finished after class.
Also, after class, Michael asked an interesting question:

Could we just integrate over the entire rectangle (square)

$R = [0,1] \times [0,1]$ and "cut the answer in half?" (Since the triangular region is half the square).

Let's see:

$$\int_{x=0}^{x=1} \int_{y=0}^{y=1} (x^2 + y^2) \, dy \, dx = \int_{x=0}^{x=1} \left[ x^2 y + \frac{y^3}{3} \right]_{y=0}^{y=1} \, dx$$

$$= \int_{x=0}^{x=1} (x^2 + \frac{1}{3}) \, dx = \left[ \frac{x^3}{3} + \frac{x}{3} \right]_{x=0}^{x=1} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

--- So apparently NOT!! ---

However, if the function $f(x,y)$ were \{symmetrical\} over the entire square region $[0,1] \times [0,1]$, then the double integral over the square would have been twice the double integral over the triangle.