NET 8: Evaluate \( \int_C (xy + z) \, ds \)

Where \( C \) consists of \( C_1 \), the line segment from \((0,0,1)\) to \((0,2,0)\) followed by the quarter-circle along \( x^2 + y^2 = 4 \) (in the xy-plane) from \((0,2,0)\) to \((2,0,0)\).

§ 17.3 Fundamental Theorem for Line Integrals, p. 1110.

Hypotheses (Preconditions) (IF's)

1. \( C \): smooth curve given by \( \vec{r}(t) \), \( a \leq t \leq b \)

2. \( f \) must be a differentiable function of two or three variables.

3. \( \nabla f \) is continuous on \( C \)

Conclusion (THEN) \( \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) \)

In General \( \int_C \vec{F} \cdot d\vec{r} \), \( \vec{F} \) is NOT \( \nabla f \) for some little \( a \).

BIG Pay-Off. Suppose that \( \vec{F} \) is conservative

(i.e., \( \exists f \exists \quad \vec{F} = \nabla f \)). Then FTLI applies

and \( \int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) \) \( \text{where } a \leq t \leq b \)

on \( C: \vec{r}(t) \).

In this case the line integral is independent of path!!

\[ \nabla f = \frac{\partial f}{\partial x} \]

\[ \frac{\partial f}{\partial y} = \frac{\partial f}{\partial x} \]

\[ \therefore \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) \]

\[ \int_C \vec{F} \cdot d\vec{r} = 0 \quad \text{for every closed path} C \quad \text{in the domain} D. \]

If \( \int_C \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} \)

for all \( C_1, C_2 \)

and since

\[ \int_{C_2} \vec{F} \cdot d\vec{r} = -\int_{C_2} \vec{F} \cdot d\vec{r} \]

it follows that \( \int_{C_0} \vec{F} \cdot d\vec{r} = 0 \).

\( \therefore \int_{C_0} \vec{F} \cdot d\vec{r} \) is independent of path, \( \text{if and only if} \quad \vec{F} \text{ is conservative} \).

If \( \vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j} \) is conservative

and if \( P,Q \) have continuous 1st order partial derivatives,

then throughout the domain \( D \):

\[ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \]
**Example:** §17.3: p. 118: #13

(a) Find \( \mathbf{F} = \langle x, y \rangle \).

(b) Use to evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \) along \( C \).

\[ C : \mathbf{T}(t) = \langle \sqrt{5} t, \sqrt{1+t^2} \rangle, \quad 0 \leq t \leq 1 \]

**Solu:**

1. Is \( \mathbf{F} \) conservative? (Cauchy Test)

   \[ P(x,y) = x^3 y^4, \quad \frac{\partial P}{\partial y} = 4x^3 y^3 \]

   and \( Q(x,y) = x^4 y^3, \quad \frac{\partial Q}{\partial x} = 4x^3 y^3 \]

   \[ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \therefore \mathbf{F} \text{ is conservative!} \]

2. It makes sense to hunt for \( f \).

   \[ \mathbf{F} = \nabla f = \langle f_x, f_y \rangle \]

   \[ \langle x^3 y^4, x^4 y^3 \rangle \quad \therefore \quad f_x = x^3 y^4 \]

   so, \( f(x,y) = \int f_x(x,y) \, dx = \int x^3 y^4 \, dx = \frac{1}{4} x^4 y^4 + g(y) \)

   \( f(y) = \frac{1}{4} x^4 y^4 + g(y) \) (※)

   But \( f_y(x,y) = x^4 y^3 + g'(y) = x^4 y^3 \quad \therefore \quad g(y) = C \) (finish at home.)
Thus, going back to (4)
\[ f(x,y) = \frac{1}{4} x^4 y^4 + C \]

Since \( \vec{F} \) is conservative, we can apply the FTLI:

(a) \( \vec{F}(t) = \langle \sqrt{t}, 1 + t^3 \rangle \); \( \vec{p}(0) = \langle 0, 1 \rangle \), \( \vec{r}(1) = \langle 1, 2 \rangle \)

(b) \[ \int_C \vec{F} \cdot d\vec{r} = \int_C \langle x^3 y^4, x^4 y^3 \rangle \cdot d\vec{r} \]
\[ = f(\vec{r}(1)) - f(\vec{p}(0)) \]
\[ = f(1, 2) - f(0, 1) \]
\[ = \frac{1}{4} (1^4 2^4) - \frac{1}{4} (0^4 1^4) \]
\[ = 4 \]

Final Summary:

[A] \( \vec{F} = x^3 y^4 \vec{i} + x^4 y^3 \vec{j} \) is conservative and its potential function is \( f(x,y) = \frac{1}{4} x^4 y^4 + C \).

[B] Using the FTLI, \( \int_C \vec{F} \cdot d\vec{r} = 4 \); \( \vec{r}(t) = \sqrt{t} \vec{i} + (1 + t^3) \vec{j} \) \( 0 \leq t \leq 1 \).