\[ \text{Revising (‡)} \]

\[ f(x,y) = \int x^4 y^4 + C \]

Now we can find \( \int_{\Gamma} \vec{F} \cdot d\vec{r} \) using FTLI.

\[ \vec{F}(t) = \left< \sqrt{t}, 1 + t^3 \right> \quad 0 \leq t \leq 1 \]

\[ \vec{F}(0) = \vec{F}(1) = \langle 0, 1 \rangle \equiv \langle 0, 1 \rangle \]

\[ \vec{F}(1) = \vec{F}(1) = \langle 1, 2 \rangle \equiv \langle 1, 2 \rangle \]

\[ \int_{\Gamma} \vec{F} \cdot d\vec{r} = \int_{\Gamma} f(x,y) \, dx + g(y) \, dy \]

\[ = \frac{1}{4} x^4 y^4 + g(y) \] (‡)

\[ \text{on the one hand} \]

\[ f_y(x,y) = x^4 y^3 \]

\[ \text{on the other hand, looking at (‡)} \]

\[ f_y(x,y) = x^4 y^3 + g'(y) \]

\[ \therefore \quad x^4 y^3 = x^4 y^3 + g(y) \]

\[ \therefore \quad g'(y) = 0 \]

\[ \therefore \quad g(y) = C \]

\[ \text{III \ §17.4 Green's Theorem, p. 118.} \]

\[ \text{Statement: 1. Let } C \text{ be a positively-oriented, piecewise-smooth, simple closed curve in the plane and let } D \text{ be the region bounded by } C. \]

\[ \text{2. If } P(x,y) \text{ and } Q(x,y) \text{ have continuous partial derivatives on an open region that contains } D, \]

\[ \text{Then } \int_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA \]
**B: Words**

- Closed curve: $a \leq t \leq b$
- Simple closed curve
- Not smooth
- Positively oriented
- $\mathbf{p}(t) = \cdots$ for $a \leq t \leq b$

**C: Problem**

- $\oint_D x^2 dy + y^3 dx$
- $C$: Rectangle with vertices $(0,0), (2,0), (2,3), (0,3)$

**Solution**

1. Sketch
2. $P(x,y) = x^2$, $Q(x,y) = y^3$
   - $P$, $Q$ have continuous partial derivatives $C$ is piecewise-smooth, positively oriented, closed, simple.

3. **Green's Theorem**

   \[
   \oint_C xy^2 dx + x^3 dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA
   \]

   - Evaluating the integrals:
     \[
     \int_{x=0}^{x=2} \int_{y=0}^{y=3} \left( 3x^2 - 2xy \right) \, dy \, dx
     \]
     \[
     = \int_{x=0}^{x=2} \left[ x^3 - x^2 y \right]_{y=0}^{y=3} \, dx
     \]
     \[
     = \int_{x=0}^{x=2} \left( 9x^2 - 9x \right) \, dx
     \]
     \[
     = \left[ \frac{3}{2}x^3 - \frac{9}{2}x^2 \right]_{x=0}^{x=2}
     \]
     \[
     = \frac{8}{3} - 2 = \frac{8}{3} - \frac{6}{3} = \frac{2}{3} = -G
     \]
§17.4: p. 1125 #13. Use Green (Check Orient.)

\[ \vec{F} = \langle \sqrt{x^3 + y^3}, x^2 + y^3 \rangle \]

Let \( C \) be a curve \( y = \sin(x) \) from \((0,0)\) to \((\pi,0)\) and the line segment from \((\pi,0)\) to \((0,0)\).

**Sketch:**

- **Orientation:** Negative

\[ \int_C \vec{F} \cdot d\vec{r} = -\int_D \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \, dA \]

Takes into account that the problem requires a negative orientation.

Class ends now... Continued after class...

3. \( \frac{\partial F_x}{\partial x} - 2x, \frac{\partial F_y}{\partial y} = 3y^2 \)

\[ \int_C \vec{F} \cdot d\vec{r} = -\int_D \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \, dA = \int_{x=0}^{x=\pi} \int_{y=0}^{y=\sin(x)} (2x^3 + y^3) \, dy \, dx \]

\[ = \int_{x=0}^{x=\pi} \left\{ 2xy - \frac{y^3}{3} \right\} \bigg|_{y=0}^{y=\sin(x)} \, dx = -\int_{x=0}^{x=\pi} \left[ 2x \sin(x) - \sin^3(x) \right] \, dx \]