Further Terminology. The study of differential equations is similar to integral calculus. When evaluating an antiderivative or indefinite integral, we still obtain a single constant of integration. In like manner, when solving a first-order differential equation \( F(x, y, y') = 0 \), we usually obtain a family of curves or points \( G(x, y, c) = 0 \) containing one arbitrary parameter such that each member of the family is a solution of the differential equation. In fact, when solving an nth-order equation \( F(x, y, y', \ldots, y^{(n)}) = 0 \), where \( y^{(n)} \) means \( d^n y/dx^n \), we express an n-parameter family of solutions \( G(x, y, c_1, \ldots, c_n) = 0 \).

A solution of a differential equation that is free of arbitrary parameters is called a particular solution. One way of obtaining a particular solution is to choose specific values of the parameter(s) in a family of solutions. For example, it is readily seen that \( y = ce^x \) is a one-parameter family of solutions of the simple first-order equation \( y' = y \). For \( c = 1 \), \( -2 \), and \( 0 \), we get the particular solutions \( y = e^x \), \( y = -2e^x \), and \( y = 0 \), respectively.

Sometimes a differential equation possesses a solution that cannot be obtained by specializing the parameters in a family of solutions. Such a solution is called a singular solution.

**Example 9** A One-Parameter Family of Solutions

In Section 2.2 we shall prove that a one-parameter family of solutions \( y' = xy^2 \) is given by \( y = (c_1 + x^2 + c_2) \). When \( c = 0 \), the resulting particular solution is \( y = x^2 \). In this case the trivial solution \( y = 0 \) is a singular solution of the equation, since it cannot be obtained from the family for any choice of parameter \( c \).

If every solution of \( F(x, y, y', \ldots, y^{(n)}) = 0 \) on an interval \( I \) can be obtained from \( G(x, y, c_1, \ldots, c_n) = 0 \) by appropriate choices of the \( c_i = 1, 2, 3, \ldots, n \), we then say that the parameter family is the general complete solution of the differential equation.

Remarks. There are two schools of thought concerning the concept of the "general solution." An alternative viewpoint holds that a general solution of an nth-order differential equation is a family of solutions containing \( n \) essential parameters. Period! In other words, the family is not required to contain all solutions of the differential equation on some interval. The difference in these opinions is really a distinction between the solutions to linear and nonlinear equations. In solving linear differential equations we shall impose relatively simple restrictions on the coefficients; with these restrictions one can always be assured not only that a solution exists but also that a family of solutions indeed yields all possible solutions.

As a matter of fact deserves mention at this time. Nonlinear equations, with the exception of some first-order equations, are usually difficult or impossible to solve in terms of familiar elementary functions such as algebraic functions, exponential and logarithmic functions, and trigonometric and inverse trigonometric functions. Furthermore, if we happen to have a family of solutions for a nonlinear equation, it is not obvious when this family constitutes a general solution. On a practical level, then, the designation "general solution" is applied only to linear differential equations.

**Exercises 1.1**

Answers to odd-numbered problems begin on page A-1.

In Problems 1–10 state whether the given differential equations are linear or nonlinear. Give the order of each equation.

1. \( (1 - x^2)y'' - 4xy' + 5y = \cos x \)
2. \( \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0 \)
3. \( y''' + 2y = 1 + x^2 \)
4. \( x^2y'' + (y' - xy')dx = 0 \)
5. \( x^3y'' - x^2y' + 4xy' - 3y = 0 \)
6. \( \frac{dy}{dx} + 9y = \sin y \)
7. \( \frac{dy}{dx} + \left[ \sqrt{1 + \frac{dy}{dx}} \right] = \frac{k}{x^2} \)
8. \( (\sin xy)'' - (\cos 2y)'' = 2 \)
9. \( (1 - x^2)dy + x dz = 0 \)

In Problems 11–40 verify that the indicated function is a solution of the given differential equation. Where appropriate, \( c_1 \) and \( c_2 \) denote constants.

11. \( y' + 2y = 0; \quad y = e^{2x} \)
12. \( y' + 4y = 32; \quad y = 8 \)
13. \( \frac{dy}{dx} - 2y = e^{-3x}; \quad y = e^{2x} + 10e^{-3x} \)
14. \( \frac{dy}{dx} + 2y = 24; \quad y = 12 - 3e^{-2x} \)
15. \( y'' + 25 + y^2; \quad y = 5 \tan 5x \)
16. \( \frac{dy}{dx} = \frac{y}{x}; \quad y = (x + c_1)^2, x > 0, c_1 > 0 \)
17. \( y'' + y = \sin x; \quad y = \frac{1}{2} \sin x - \frac{1}{4} \cos x + 10x + 2 \)
18. \( 2dy + (x^2 + 2y)dx = 0; \quad y = \frac{1}{x} \)
19. \( (y')^2 + x^2 dy = 0; \quad y = \frac{-1}{x} \)
20. \( (y')^2 + xy = y; \quad y = x + 1 \)
21. \( y = 2xy' + x(y')^2; \quad y' = c_1(x + 1/c_1) \)
22. \( y'' = 2\sqrt{y}; \quad y = x^4 \)

23. \( y' + x^2 = 1; \quad y = x \ln x, x > 0 \)

24. \( \frac{dp}{dt} = P(a - bP); \quad P = \frac{\ln w}{1 + cxe^y} \)

25. \( \frac{dx}{dt} = (2 - x)(1 - x); \quad \ln 2 - \frac{x}{1 - x} = t \)

26. \( y' + 2xy = 1; \quad y = e^{-x} \int 0^x e^{2t} dt + c_1 e^{-x} \)

27. \( (x^2 + y^2) \frac{dx}{dt} + (x^2 - xy) \frac{dy}{dx} = 0; \quad c_1 x + y^2 = x e^{2x} \)

28. \( y' + 15y = 0; \quad y = c_1 e^{5x} + c_2 e^{-x} \)

29. \( y' - 2y' - 6y + 13y = 0; \quad y = e^x \cos 2x \)

30. \( \frac{dy}{dx^2} - 4 \frac{dy}{dx} + 4y = 0; \quad y = e^{2x} + xe^{2x} \)

31. \( y' = y; \quad y = c_1 \cos x + c_2 \sin x \)

32. \( y' + 25y = 0; \quad y = c_1 \cos 5x \)

33. \( y'' + (y')^2 = 0; \quad y = \ln |x + c_1| + c_2 \)

34. \( y'' + y = \tan x; \quad y = -c_1 \ln(\sec x + \tan x) \)

35. \( x^2 \frac{dy}{dx} + y = 0; \quad y = c_1 + c_2 x^{-1} \)

36. \( x^2 \frac{dy}{dx} - 2y = 0; \quad y = c_1 x \ln x, x > 0 \)

37. \( x^2 \frac{dy}{dx} - 3x y' + 4y = 0; \quad y = x^2 + c_1 \ln x, x > 0 \)

38. \( y'' - y' + 9y' = 0; \quad y = c_1 \sin 3x + c_2 \cos 3x + 4e^x \)

39. \( y'' - 3y' - 4y = 0; \quad y = c_1 e^{2x} + c_2 e^{-2x} \)

40. \( x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + y' = 12e^x; \quad y = c_1 + c_2 x + c_3 \ln x + 4x^2, x > 0 \)

In Problems 41 and 42 verify that the indicated piecewise-defined function solution of the given differential equation.

41. \( y'' - 2y' - y = 0; \quad \begin{cases} y = \begin{cases} x^2, & x < 0 \\ x^3, & x > 0 \end{cases} \\ y' = \begin{cases} 2x, & x < 0 \\ 3x^2, & x > 0 \end{cases} \end{cases} \)

42. \( (y')^2 = 9x; \quad \begin{cases} y = \begin{cases} 0, & x < 0 \\ x^2, & x > 0 \end{cases} \\ y' = \begin{cases} 0, & x < 0 \\ 2x, & x > 0 \end{cases} \end{cases} \)

43. Verify that a one-parameter family of solutions for

\( y = xy' + (y')^2 \quad \text{is} \quad y = cx + y^2 \)

Determine a value of \( c \) such that \( y = \frac{1}{2}x^2 \) is a singular solution of the differential equation.

44. Verify that a one-parameter family of solutions for

\( y = xy' + \sqrt{x + (y')^2} \quad \text{is} \quad y = cx + \sqrt{1 + c^2} \)

Show that the relation \( x^2 + y^2 = 1 \) defines a singular solution of the equation on the interval \((-1, 1)\).
Dividing the last two equations, we find
\[ \tan \theta = \frac{w_2}{T_2} \text{ or } \frac{dy}{dx} = \frac{w_2}{T_2}. \]

Now since the length of the arc between points \( P_1 \) and \( P_2 \) is
\[ s = \int \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx, \]
it follows from one form of the fundamental theorem of calculus that
\[ ds \frac{d}{dx} = \sqrt{1 + \left( \frac{dy}{dx} \right)^2}. \]

Differentiating (12) with respect to \( x \) and using (13) lead to
\[ \frac{d^2y}{dx^2} = \frac{w}{T_1} \frac{ds}{dx} \text{ or } \frac{d^2y}{dx^2} = \frac{w}{T_1} \sqrt{1 + \left( \frac{dy}{dx} \right)^2}. \] (14)

One might conclude from Figure 1.10 that the shape the hanging wire assumes is parabolic. However, this is not the case; a wire or heavy rope hank under only its own weight takes the shape of a hyperbolic cosine. See Problem Exercises 5.3. Recall that the graph of the hyperbolic cosine is called a catenary which stems from the Latin word catedra meaning "chain." The Romans used the catenary as a dog leash. Probably the most graphic example of the shape of a catenary is the 630-ft-high Gateway Arch in St. Louis, Missouri.

Discharge Through an Orifice In hydrodynamics, Torricelli’s theorem is that the speed \( v \) of efflux of water through a sharp-edged orifice at the base of a tank filled to a depth \( h \) is the same as the speed that a body (in this case drop of water) would acquire in falling freely from a height \( h \):
\[ v = \sqrt{2gh}, \]
where \( g \) is the acceleration due to gravity. This last expression comes from equating the kinetic energy \( \frac{1}{2}mv^2 \) with the potential energy \( mgh \) and solving for \( v \).

### Depth of Water in a Draining Tank

Suppose a tank filled with water is allowed to discharge through an orifice under the influence of gravity. We would like to find the depth \( h \) of water remaining in the tank at time \( t \).

Consider the tank shown in Figure 1.11. If the area of the orifice \( A_o \) (in \( \text{in}^2 \)) and the speed of the water leaving the tank is \( v = \sqrt{2gh} \) (in \( \text{ft} \)), then the volume of water leaving the tank per second is \( A_o \sqrt{2gh} \) (in \( \text{ft}^3 \)). Thus, if \( V(t) \) denotes the volume of water in the tank at time \( t \), then
\[ \frac{dV}{dt} = A_o \sqrt{2gh}, \] (15)

where the minus sign indicates that \( V \) is decreasing. Note here that we are ignoring the possibility of friction at the orifice, which might reduce the rate of flow there.

Now if the tank is such that the volume of water in it at time \( t \) can be written as \( V(t) = A_o h \), where \( A_o \) (in \( \text{ft}^2 \)) is the constant area of the upper surface of the water (see Figure 1.11), then \( dV/dt = A_o (dh/dt) \). Substituting this last expression into (15) gives us the desired differential equation for the height \( h \) of the water:
\[ \frac{dh}{dt} = -\frac{A_o}{A_w} \sqrt{2gh}. \] (16)

It is interesting to observe that (16) remains valid even when \( A_o \) is not constant. In this case we must express the upper surface area of the water as a function of \( h \): \( A_w = A(h) \). See Problem 9 in Exercises 1.2 and Problem 19 in the Chapter 1 Review Exercises.

### Deflection of Beams

In engineering an important problem is to determine the static deflection of an elastic beam caused by its weight or by an external load. We assume that the beam is homogeneous and has uniform cross sections along its length. Let \( L \) denote the length of the beam. In the absence of any load on the beam (including its weight), a curve joining the centroids of all its cross sections is a straight line called the axis of symmetry. See Figure 1.12(a). If a load is applied to the beam in a vertical plane containing the axis of symmetry, then, as shown in Figure 1.12(b), the beam undergoes a distortion and the curve connecting the centroids of all cross sections is called the deflection curve or elastic curve. In the next example we derive the differential equation of the deflection curve. This derivation uses principles from elasticity and a concept from calculus called curvature.

#### Deflection of a Cantilever Beam

For the sake of illustration let us consider a cantilever beam embedded at its left end and free at its right end. As shown in Figure 1.13, we let the embedded end of the beam coincide with \( x = 0 \) and its free end with \( x = L \). The \( x \)-axis coincides with the axis of symmetry, and the deflection \( y(x) \) is measured from this axis and is considered positive if downward. Now in the theory of elasticity it is shown that the bending moment \( M(x) \) at a point \( x \) along the beam is related to the load per unit length \( w(x) \) by the equation
\[ \frac{d^2M}{dx^2} = w(x). \] (17)
where \( r \) is the annual rate of interest. This mathematical description is analogous to the population growth of Example 10. The rate of growth is greater when the amount of money present in the account is also large. Geometrically, this means the tangent line is steep when \( S \) is large. Figure 1.14.

The definition of a derivative provides an interesting derivation of (Suppose \( S(t) \) is the amount of money accrued in a savings account after \( t \) years when the annual rate of interest \( r \) is compounded continuously. If \( k \) denotes an increment in time, then the interest obtained in the time span \((t + \Delta t) - t\) is

\[
S(t + \Delta t) - S(t)
\]

Since interest is given by (rate) \( \times \) (time) \( \times \) (principal), we can approximate the interest earned in this same time period by either

\[
rh S(t) \quad \text{or} \quad rS(t + \Delta t)
\]

Intuitively we see that \( rS(t) \) and \( rS(t + \Delta t) \) are lower and upper box respectively, for the actual interest (27); that is,

\[
rS(t) + rS(t + \Delta t) - S(t - S(t + \Delta t)) \leq rS(t + \Delta t)
\]

or

\[
rS(t) \leq \frac{S(t + \Delta t) - S(t)}{\Delta t} \leq rS(t + \Delta t)
\]

Taking the limit of (28) as \( \Delta t \to 0 \), we get

\[
rS(t) = \lim_{\Delta t \to 0} \frac{S(t + \Delta t) - S(t)}{\Delta t} = rS(t + \Delta t)
\]

and so it must follow that

\[
\lim_{\Delta t \to 0} \frac{S(t + \Delta t) - S(t)}{\Delta t} = rS(t)
\]

or

\[
\frac{dS}{dt} = rS
\]

EXERCISES 1.2

Answers to odd-numbered problems begin on page A-1.

In Problems 1–22 derive the differential equation(s) describing the given physical situation:

1. Under some circumstances a falling body \( B \) of mass \( m \), such as the sky shown in Figure 1.15, encounters air resistance proportional to its instantaneous velocity \( v \). Use Newton's second law to find the differential equ for the velocity \( v \) of the body at time \( t \). Recall that accelerating a body \( \frac{dv}{dt} \). Assume in this case that the positive direction is downward.

2. What is the differential equation for the velocity \( v \) of a body of mass falling vertically downward through a medium (such as water) that offers resistance proportional to the square of its instantaneous velocity? Assume the positive direction is downward.

3. By Newton's universal law of gravitation the free-fall acceleration \( a \) of a body, such as the satellite shown in Figure 1.16, falling a great distance to the surface is not the constant \( g \). Rather, the acceleration \( a \) is inversely proportional to the square of the distance \( r \) from the center of the earth, \( a = k/r^2 \), where \( k \) is the constant of proportionality.

(a) Use the fact that at the surface of the earth \( r = R \) and \( a = g \) to determine the constant of proportionality \( k \).

(b) Use Newton's second law and part (a) to find a differential equation for the distance \( r \).

(c) Use the chain rule in the form

\[
\frac{dv}{dt} = \frac{dv}{dr} \cdot \frac{dr}{dt}
\]

to express the differential equation in part (b) as a differential equation involving \( v \) and \( dv/dr \).

4. (a) Use part (b) of Problem 3 to find the differential equation for \( r \) if the resistance to the falling satellite is proportional to its instantaneous velocity.

(b) Near the surface of the earth, use the approximation \( R = r \) to show that the differential equation in part (a) reduces to the equation derived in Problem 1.

5. A series circuit contains a resistor and an inductor as shown in Figure 1.17. Determine the differential equation for the current \( i(t) \) if the resistance is \( R \), the inductance is \( L \), and the impressed voltage is \( E(t) \).

6. A series circuit contains a resistor and a capacitor as shown in Figure 1.18. Determine the differential equation for the charge \( q(t) \) on the capacitor if the resistance is \( R \), the capacitance is \( C \), and the impressed voltage is \( E(t) \).

7. Suppose a tank is discharging water through a circular orifice of cross-sectional area \( A_0 \) at its bottom. It has been shown experimentally that when water leaves the tank per second is approximately \( 0.64A_0 \sqrt{2gh} \). Find the differential equation for the height \( h \) of water at time \( t \) for the cubical tank in Figure 1.19. The radius of the orifice is \( 2 \) in. and \( g = 32 \) ft/s².

8. A tank in the form of a right circular cylinder of radius 2 ft and height 10 ft is standing on end. The tank is initially full of water, and water leaks from a circular hole of radius \( \frac{1}{4} \) in. at its bottom. Use the information in Problem 7 to obtain the differential equation for the height \( h \) of the water at time \( t \).

9. A water tank has the shape of a hemisphere with radius 5 ft. Water leaks out of a circular hole of radius 1 in. at its flat bottom. Use the information in Problem 7 to obtain the differential equation for the height \( h \) of the water at time \( t \).

10. The rate at which a radioactive substance decays is proportional to the amount \( A(t) \) of the substance remaining at time \( t \). Determine the differential equation for the amount \( A(t) \).

11. A drug is infused into a patient's bloodstream at a constant rate of 1 gram per second. Simultaneously, the drug is removed at a rate proportional to the amount \( x(t) \) of the drug present at time \( t \). Determine the differential equation governing the amount \( x(t) \).
12. A projectile shot from a gun has weight $w = mg$ and velocity $v$ along its path of motion. Ignoring air resistance and all other forces, find the system of differential equations that describes the See Figure 1.20. [Hint: Use Newton’s second law in the $x$ and $y$ directions.]

13. Determine the equations of motion if the projectile in Problem 12 on a rotating body $k$ (of magnitude $k$) acting tangent to the path but not the motion. See Figure 1.21. [Hint: $k$ is a multiple of the velocity, $v$.]

14. Two chemicals $A$ and $B$ react to form a new chemical $C$. Assume the concentrations of both $A$ and $B$ decrease by the amount formed, find the differential equation governing the concentration of chemical $C$ at the rate at which the chemical reaction takes place is proportional to the product of the remaining concentrations and $B$.

15. Light strikes a plane curve $C$ in such a manner that all beams $L$ per the $y$-axis are reflected to a single point $O$. Determine the differential equation for the function $y = f(x)$ describing the shape of the curve, (that the angle of incidence is equal to the angle of reflection is $a_1$ of optics.) [Hint: Use Inspectors of Figure 1.22 and 1.23 shows that the inclination tangent line from the horizontal at $P(x, y)$ is $y' = \theta$ and that $\theta = \phi - 2\theta$. (Why?)] Also, don’t be afraid to use a trigonometric identity.

16. A cylindrical barrel $r$ feet in diameter of weight $w$ lb is $k$ feet water. After an initial depression, the barrel exhibits an up-and-down bung motion along a vertical line. Using Figure 1.23(a), determine the differential equation for the vertical displacements $y(t)$ if the origin is $b$ on the vertical axis at the surface of the water when the barrel is taken to be on the vertical axis at the surface of the water displaced and the density of water is $2.64 \text{ lb/ft}^3$. Assume that the downward is positive. Ignore the resistance of the water.

17. A rocket is shot vertically upward from the surface of the earth. At fuel has been expended, the mass of the rocket is a constant $m$. Newton’s second law of motion and the fact that the force of gravity varies as the square of the distance to find the differential equation: $y$ from the earth’s center to the rocket at time $t$ after burn appropriate conditions with this differential equation.

18. Newton’s second law $F = ma$ can be written $F = d \left( \frac{mv^2}{2} \right)$. When of an object is variable, this latter formulation is used. The mass rocket launched upward changes as its fuel is consumed. If $v(t)$ is velocity at any time, it can be shown that

$$-mg = m \frac{dv}{dt} = \sqrt{2a^2 v^2}. \tag{39}$$

* It is assumed that the total mass, mass of vehicle + mass of fuel + mass of exhaust gases, is constant. In this case $m(t) = \text{mass of vehicle} + \text{mass of fuel}$. where $V$ is the constant velocity of the exhaust gases relative to the rocket. Use (29) to find the differential equation for $v$ if it is known that $m(t) = m_0 - Vt - b$, where $m_0$, $a$, and $b$ are constants.

19. A person $P$, starting at the origin, moves in the direction of the positive $x$-axis, pulling a weight along the curve $C$ (called a tractrix) as shown in Figure 1.24. The weight, initially located on the $y$-axis at $(0, y)$, is pulled by a rope of constant length $s$, which is kept taut throughout the motion. Find the differential equation of the path of motion. [Hint: The rope is always tangent to $C$; consider the angle of inclination $\theta$ as shown in the figure.]

20. Suppose a hole is drilled through the center of the earth. A body with mass $m$ is dropped into the hole. Let the distance from the center of the earth to the mass at time $t$ be denoted by $r$. See Figure 1.25.

(a) Let $M$ denote the mass of the earth and $M'$ denote the mass of that portion of the earth within a sphere of radius $r$. The gravitational force on $m$ is $F = -kMm/r^2$, where the minus sign indicates that the force is one of attraction. Use this fact to show that

$$F = \frac{kMm}{r^2}. \tag{41}$$

* [Hint: Assume that the earth is homogeneous—that is, has a constant density $\rho$. Use mass $= \text{density} \times \text{volume}.]

(b) Use Newton’s second law and the result in part (a) to derive the differential equation

$$\frac{d^2r}{dt^2} + \frac{kr^2}{M'} = 0,$$

where $M' = kM/R^2 = r$. (Why?)

* 21. In the theory of learning, the rate at which a subject is memorized is assumed to be proportional to the amount that is left to be memorized. If $M$ denotes the total amount that is to be memorized and $A(t)$ the amount memorized in time $t$, find the differential equation for $A$.

* 22. In Problem 21 assume that the amount of material forgotten is proportional to the amount memorized in time $t$. What is the differential equation for $A$ when forgetfulness is taken into account?

CHAPTER 1 REVIEW

We classify a differential equation by its type, ordinary or partial; by its order; and by whether it is linear or nonlinear. A solution of a differential equation is any function having a sufficient number of derivatives that satisfies the equation identically on some interval. When solving an nth-order ordinary differential equation, we expect to find an $n$-parameter family of solutions. A particular solution is any solution free