§ 4.1.3: p. 120 - Superposition principle for Homog. Eq.'s. (HLODEs)

A Example - Given \( y'' + 3y' + 2y = 0 \) (*)

If I tell you that \( y_1 = e^{-x} \) is a solution to (*) and if I tell you that \( y_2 = e^{-2x} \) is a solution to (*), then by the superposition principle,

\[ y_3 = 7.5e^{-x}, \quad y_4 = \sqrt{\pi}e^{-2x}, \quad y_5 = e^{-x} - 14e^{-2x} \]

etc, etc, etc — are all solutions to (*).

In fact, every possible solution to (*) is of the form \( y = c_1e^{-x} + c_2e^{-2x} \)

B Be advised — what the S.P. guarantees is that If \( y = c_1e^{-x} + c_2e^{-2x} \) (for any \( c_1, c_2 \in \mathbb{R} \)),

then \( y \) is a solution to (*).

C We need something more before we can say that the converse is true (i.e. If \( y \) is a sol. to (*), then \( y \) has the form \( y = c_1e^{-x} + c_2e^{-2x} \) for some choice of \( c_1, c_2 \).

II Th. 4.4: p. 122: If \( y_1, \ldots, y_n \) are \( n \)-sols. to an \( n \)-th order HLODE on \( I \), then the set \( \{ y_1, y_2, \ldots, y_n \} \) is LI \( \iff \) \( W(y_1, y_2, \ldots, y_n)(x) \neq 0 \) \( \forall x \in I \)

\( \iff \) and only if
Def 4.3 p 123 — Fundamental Set of Solutions

If \{y_1, y_2, \ldots, y_n\} is a LI set of n-solutions to an n-th order HLODE on an interval \(I\), then the set \{y_1, y_2, \ldots, y_n\} is said to be a FUNDAMENTAL SET OF SOLUTIONS on the interval \(I\).

**BLOCKBUSTER — THM. 4.5**

If \{y_1, \ldots, y_n\} is a fund. set. of sols to an n-th order HLODE, then if \(Y(x)\) is any sol. to the HLODE, then

\[ Y(x) = C_1y_1 + C_2y_2 + \ldots + C_ny_n = \sum_{i=1}^{n} C_iy_i(x) \]

**Thm 4.6 — Every n-th order HLODE has a Fund. Set of Sols.**


\[ a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = g(x) \]  (***)

Any function \(y_p(x)\) which is free of arbitrary parameters \((c_i)'s\) which satisfies (***) is called a PARTICULAR SOLUTION to (***)

**Thm 4.7.** If \{y_1, y_2, \ldots, y_k\} are sols. of an n-th order HLODE and if \(y_p\) is a particular sol. to the corresponding non-HLODE, then
\[ y = c_1 y_1 + c_2 y_2 + \ldots + c_k y_k + y_p \quad \text{is a solution to the non-HLODE.} \]

\[ \text{Thm 4.8 p 126} \]

\[ \text{If } y_p \text{ is a part. sol. to a non-HLODE (n-th order) and if } \{ y_1, \ldots, y_n \} \text{ is a fund. set of sols to the corresponding HLODE on } I, \]

\[ \text{Then } Y = c_1 y_1 + \ldots + c_n y_n + y_p \quad (\#) \quad \text{is a sol to the non-HLODE and EVERY sol to the non-HLODE has the form (\#) for some } c_i \quad i=1\ldots n. \]

\[ \text{Example. (Based Upon Ex 25, p. 127).} \]

\[ \text{Given } y''' - 6y'' + 11y' - 6y = 3x \quad \text{NON-HLODE.} \]

\[ \text{Related } y''' - 6y'' + 11y' - 6y = 0 \]

\[ \text{Given } y_c = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} \text{ is a sol to the HLODE on } (-\infty, \infty). \]

\[ \text{Given } y_p = -\frac{11}{12} - \frac{1}{2} x \text{ is a part. sol. to the non-HLODE.} \]

\[ \text{Then } y = y_c + y_p = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} - \frac{11}{12} - \frac{1}{2} x \text{ is a sol. (Gen Sol) to the non-HLODE.} \]

\[ \text{Tomorrow we'll start using some of this stuff in § 4.3. We are temporarily skipping § 4.2.} \]