§ 4.3 — Case III Aux Poly is prime.
A Example: \( y'' - 4y' + 7y = 0 \) (*)

1 AuxEq. \( m^2 - 4m + 7 = 0 \)
2 Com Sq. \( m^2 - 4m + 4 = -7 + 4 \)
   \( (m-2)^2 = -3 \)
   \( m-2 = \pm \sqrt{-3} = \pm i \sqrt{3} \)
   \( m = 2 \pm i \sqrt{3} \)

By all "rights" the sol should be

\( y = c_1 e^{(2+i\sqrt{3})x} + c_2 e^{(2-i\sqrt{3})x} \)

B How do we eliminate "i"? \( i = \sqrt{-1} \)

1 Euler's equation: \( e^{i \theta} = \cos(\theta) + i \sin(\theta) \)
2 "Notation \( \text{cis}(\theta) = \cos(\theta) + i \sin(\theta) \)"
3 Aside: \( e^{i\pi} = \cos(\pi) + i \sin(\pi) = -1 \)
   \( e^{i\pi} = -1 \) \( \leftarrow \) Voted #1 "Most Beautiful Eqn" in all math!

4 Back to the problem — in General
   If \( m = \alpha \pm \beta i \) are the roots of the Aux Eq.
   then
   
   (cont...)
\[ y = c_1 e^{(\alpha + \beta i)x} + c_2 e^{(\alpha + \beta i)x} \quad (\star) \]

b. Look at \( e^{(\alpha \pm \beta i)x} = e^{(\alpha x \pm \beta ix)} \)

\[ = e^{\alpha x} (\pm \beta ix) \quad \text{focus on this factor.} \]

\[ e^{\beta ix} = e^{i\beta x} = (\cos(\beta x) + i\sin(\beta x)) \]
\[ e^{-\beta ix} = e^{i(-\beta x)} = \cos(-\beta x) + i\sin(-\beta x) \]
\[ = \cos(\beta x) - i\sin(\beta x) \]

c. \[ e^{\beta ix} + e^{-\beta ix} = 2\cos(\beta x) \quad (\star\star) \]

\[ e^{\beta ix} - e^{-\beta ix} = 2i\sin(\beta x) \]

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**END of class // After class**

d. Since (\star) is the Gen.Sol, any choice for \( c_1 \land c_2 \) yields a sol.

\[ y_1 = e^{(\alpha + \beta i)x} + e^{(\alpha - \beta i)x} \]
\[ \text{is a sol} \quad [c_1 = c_2 = 1] \quad \text{and} \]
\[ y_2 = e^{(\alpha + \beta i)x} - e^{(\alpha - \beta i)x} \quad \text{or} \quad [c_1 = 1, \quad c_2 = -1]. \]

e. But \( y_1 = e^{\alpha x} (e^{i\beta x} + e^{-i\beta x}) = 2e^{\alpha x}\cos(\beta x) \)

\[ y_2 = e^{\alpha x} (e^{i\beta x} - e^{-i\beta x}) = 2i e^{\alpha x}\sin(\beta x) \quad \text{by (\star\star)} \]

f. But a "constant-times-a-solution" is also a solution; therefore,

\[ \hat{y}_1 = \frac{1}{2} y_1 = e^{\alpha x}\cos(\beta x) \]
\[ \hat{y}_2 = (-\frac{1}{2} i) y_2 = e^{\alpha x}\sin(\beta x) \]

are also solutions.

In Summary \[ \hat{y}_1 = e^{\alpha x}\cos(\beta x) \quad \hat{y}_2 = e^{\alpha x}\sin(\beta x) \quad (\star\star\star) \]
9. Now the next thing we have to do is to show that $\hat{y}_1$ and $\hat{y}_2$ are L.I. (Because what we're aiming for is to show that $y = c_1 \hat{y}_1 + c_2 \hat{y}_2$ is the GenSol.)

Since we know that $\hat{y}_1$ and $\hat{y}_2$ are sols to a HLODE w/const. coef.,
we can use an "iff" thm. to show $\hat{y}_1, \hat{y}_2$ are L.I. $\iff W \neq 0 \forall x \in I$.

Proof: Prelim. work, $\hat{y}_1' = (e^{ax} \cos(\beta x))'$

$= -\beta e^{ax} \sin(\beta x) + \alpha e^{ax} \cos(\beta x) = e^{ax}(\alpha \cos(\beta x) - \beta \sin(\beta x))$

and

$\hat{y}_2' = (e^{ax} \sin(\beta x))' = \beta e^{ax} \cos(\beta x) + \alpha e^{ax} \sin(\beta x)$

$= e^{ax}(\beta \cos(\beta x) + \alpha \sin(\beta x))$

Wronskian:

$W(\hat{y}_1, \hat{y}_2)(x) = \begin{vmatrix} e^{ax} \cos(\beta x) & e^{ax} \sin(\beta x) \\ e^{ax}(\alpha \cos(\beta x) - \beta \sin(\beta x)) & e^{ax}(\beta \cos(\beta x) + \alpha \sin(\beta x)) \end{vmatrix}$

$= e^{2ax} \left( \beta \cos^2(\beta x) + \alpha \sin(\beta x) \cos(\beta x) \right)$

$- e^{2ax} \left( \alpha \sin(\beta x) \cos(\beta x) - \beta \sin^2(\beta x) \right)$

$= e^{2ax} \left( \beta \cos^2(\beta x) + \alpha \sin(\beta x) \cos(\beta x) - \alpha \sin(\beta x) \cos(\beta x) + \beta \sin^2(\beta x) \right)$

$= \beta e^{2ax} \neq 0 \ \forall x \in (-\infty, \infty)$ \hspace{1cm} \[Assuming \ \beta \neq 0, \ i.e. \ assuming \ that \ we \ are \ actually \ dealing \ with \ \alpha \pm \beta i \ NON-REAL! \]

\[\therefore e^{ax} \cos(\beta x) \ and \ e^{ax} \sin(\beta x) \ are \ L.I. \ on \ (-\infty, \infty).\]

Thus $\{e^{ax} \cos(\beta x), e^{ax} \sin(\beta x)\}$ is a \textbf{FUNDAMENTAL} set of solutions on $(-\infty, \infty)$. 
Consequently: $y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$ is the GENSOL to the HLODE w/ const. coef.

Now we can actually go back to EXAMPLE A above. Let's resolve the problem from start to finish.

1. Solve: $y'' - 4y' + 7y = 0$
   
   Soln: 1. AuxEq. $m^2 - 4m + 7 = 0$
   
   2. Solve AuxEq. $m^2 - 4m + 4 = -7 + 4 = -3$
      
      $(m - 2)^2 = -3$
      
      $m = 2 \pm i\sqrt{3}$ < non-real
   
   3. GenSol: $y = c_1 e^{2x} \cos(\sqrt{3}x) + c_2 e^{2x} \sin(\sqrt{3}x)$

   and that's all there is to it!

Let me do one more, just to fill out the page:

Solve $y'' + 2y' + 3y = 0$

Soln: 1. AuxEq. $m^2 + 2m + 3 = 0$
   
   2. Solve AuxEq. $m^2 + 2m + 1 = -3 + 1$
      
      $(m + 1)^2 = -2$
      
      $m = -1 \pm i\sqrt{2}$
   
   3. GenSol: $y = c_1 e^{-x} \cos(\sqrt{2}x) + c_2 e^{-x} \sin(\sqrt{2}x)$