5.4.5: Section. Annihilator Operator. (Wipe-Out)

P. 159.

A. Most every function that we deal with can be thought of as the solution to some higher order linear DE with constant coefficients. (Jeopardy for ODE's)

B. Examples:

1. \( y = 8x^2 + 3x - 1 \) is a function. Give me a Diff. Op. \( L \) such that \( L[y] = 0 \)

   \[ D^3(y) = 0 \text{ b/c.} \]
   \[ y''' = 0 + 0 - 0 = 0 \]

   \[ \boxed{ D^3 \text{ Annihilates } y_1 = 8x^2 + 3x - 1} \]

2. \( y_2 = e^x \). What Diff. Op. Annihilates \( y_2 \)?

   \[ D^2 - 1 \text{ Think } y_2' - y_2 = 0 \ldots \text{ so in Diff. Op. notation} \]

   Mental Check: \( D - 1)(e^x) = De^x - e^x = e^x - e^x = 0 \)

   \[ \boxed{ D - 1 \text{ Annihilates } e^x} \]

3. Annihilate \( y_3 = \sin(2x) \). Solve:

   \[ D^2 + 4 \ldots \text{ check,} \]

   \[ (D^2 + 4)(\sin(2x)) = D^2\sin(2x) + 4\sin(2x) \]
   \[ = -4\sin(2x) + 4\sin(2x) = 0 \]

   \[ \boxed{ \text{Annihilates } \sin(2x)} \]
4. Annihilate \( y_4 = 2e^x + e^{3x} \)

Solve: Think... \( (D-1)(2e^x) = D(2e^x) - (2e^x) = 2e^x - 2e^x = 0 \)

And \( (D-3)(e^{3x}) = D(e^{3x}) - 3e^{3x} = 3e^{3x} - 3e^{3x} = 0 \)

So... Hmmmm... Look at the Diff. Ops in series.

\[
\left[(D-1)(D-3)\right](2e^x + e^{3x}) = (D-1)\left[(D-3)(2e^x + e^{3x})\right] = (D-1)\left[D(2e^x + e^{3x}) - 3(2e^x + e^{3x})\right]
\]

\[
= (D-1)\left[2e^x + 3e^{3x} - 6e^x - 3e^{3x}\right]
\]

\[
= (D-1)(-4e^x) = D(-4e^x) - (-4e^x) = -4e^x + 4e^x = 0
\]

So \( L = (D-1)(D-3) \) annihilates \( y_4 = 2e^x + e^{3x} \)

5. Note: \( y_4 = 2e^x + e^{3x} \) is a sol. to
\[
y'' - 4y' + 3y = 0
\]

II § 4.6: Solving Non-HLODE w/const. coef. By UNDETERMINED COEFFICIENTS (UC) - Ann. Approach

(UCAA)
S 4.6: p. 168: #11. Solve using UCAA.
\[ y'' - 2y' - 3y = 4e^x - 9 \quad (\ast) \]

**Solution**

1. Find \( y_c \). \( y'' - 2y' - 3y = 0 \)
   \[ m^2 - 2m - 3 = 0 \quad \text{AuxEq.} \]
   \[ (m-3)(m+1) = 0 \rightarrow m_1 = 3, \; m_2 = -1 \]
   \[ y_c = c_1 e^{3x} + c_2 e^{-x} \quad (***) \]

2. Ann. the forcing function \( g(x) = 4e^x - 9 \)
   \[ \text{Ann. } D-1 \text{ ann. } 4e^x \quad \text{and} \quad D \text{ Ann. } (-9) \]
   \[ \therefore D(D-1) \text{ Annihilates } g(x). \]
   Let \( L_1 = D(D-1) \)

3. Put \((\ast)\) into DiffOp form
   \[ (D^2 - 2D - 3)(y) = 4e^x - 9 \]
   Let \( L = D^2 - 2D - 3 = (D-3)(D+1) \)

4. Consider \( L_1 L(y) \)
   \[ L(y) = 4e^x - 9 \]
   \[ L_1 L(y) = L_1(4e^x - 9) \]
   \[ D(D-1)(D-3)(D+1)(y) = D(D-1)(4e^x - 9) = 0 \]

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**Class Ends — After Class**

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This yields the 4th order HLODE w/const. coef whose

\[ \text{AuxEq. factors into } m(m-1)(m-3)(m+1) = 0 \]
Thus the GenSol to
\[ y^{(4)} - 3y''' - y'' + 3y' = 0 \]
is \[ \hat{y} = c_1 e^{ox} + c_2 e^{x} + c_3 e^{3x} + c_4 e^{-x} \]
or \[ \hat{y} = c_1 + c_2 e^{x} + c_3 e^{3x} + c_4 e^{-x} \] (***)

Now WITHIN (***), we "see" \( y_c \) — (***)
So if we discard \( y_c \) from \( \hat{y} \), we have
\[ \hat{y} = c_1 + c_2 e^{x} \]
(The subscripts on the constants do not have to "match.")
So the CLAIM is that FOR SOME PARTICULAR, BUT AS OF YET UNKNOWN CONSTANTS, we have found one particular soln. to (**). Thus
\[ y_p = A + Be^{x} \] (A) (we use \( A \) & \( B \) rather than \( c_1 \) and \( c_2 \) because \( A \) & \( B \) are specific — but as of now unknown constants; whereas, \( c_1 \) & \( c_2 \) represent arbitrary constants)

Now all we have to do is find what \( A \) & \( B \) really are.

To find, indeed, \( y_p \) solves (**), we can subs and get an identity.

i. \( y_p' = Be^{x} \) \( \Rightarrow \) \( y_p'' = Be^{x} \) (This was easier than usual!)

Assumption

ii. \[ 4e^{x} - 9 = y_p'' - 2y_p' - 3y_p = (Be^{x}) - 2(Be^{x}) - 3(A + B) = (B - 2B - 3B)e^{x} - 3A \]
\[ \therefore -4B = 4 \text{ and } -3A = -9 \Rightarrow B = -1 \text{ and } A = 3 \]
\[ \therefore \] (\( y_p = 3 - e^{x} \) (***))
Therefore the GenSol to (*)

\[ y'' - 2y' - 3y = 4e^x - 9 \]

is

\[ y = y_c + y_p = c_1 e^{3x} + c_2 e^{-x} + 3 - e^x \]