§ 4.5: DIFFERENTIAL OPERATORS, p. 157

A. Very Simple

If \( y = f(x) \) is a function, then
\[
D[y] = \frac{dy}{dx}
\]

\[
D[x^2] = 2x
\]

B. "D" is known as a differential operator

C. Main Property
\[
D[\alpha f(x) + \beta g(x)] = \alpha D[f(x)] + \beta D[g(x)]
\]
Linear Property.
So D is a linear operator.

D. Is "taking the integral" a linear operation?
\[
\int [\alpha f(x) + \beta g(x)] \, dx = \alpha \int f(x) \, dx + \beta \int g(x) \, dx
\]
Looks linear.

Example
\[
\int (2x + 3x^2) \, dx = x^2 + x^3 + C
\]
2 \( \int x \, dx + 3 \int x^2 \, dx = (x^2 + C) + (x^3 + C)
\]
= \( x^2 + x^3 + C \)

E. Back to the "Big D."

Consider the ODE \( y'' + 3y' + 2y = 0 \) \((*)\)

I want to learn how to write \((*)\) in operator
notation.

\[ D^2[y] = y'' \]

\[ D^2[y] + 3D[y] + 2y = 0 \]

Form the "compound operator"

\[ (D^2 + 3D + 2)[y] = 0 \quad (***) \]

LHS means \( y'' + 3y' + 2 \)

There is another operator here — namely the "identity operator", \( I \) — where

\[ I[y] = y \quad I[\alpha y] = \alpha I[y], I[y_1 + y_2] = \alpha I[y_1] + \beta I[y_2] = \alpha y_1 + \beta y_2 \]

So (***) could have been written

\[ (D^2 + 3D + 2I)[y] = 0 \]

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**Examples. — Put into op. notation**

1. \( 4y'' - 7y' + 3y = \sin(x) \)

   **Solution:** \((4D^3 - 7D + 3)[y] = \sin(x)\)

2. \( xy' + y = \cos(x) \)

   **Solution:** \((xD + 1)[y] = \cos(x)\)
4. Reconsider \( y'' + 3y' + 2y = 0 \) (*).

1. Keep in mind the "old" (§4.3) method of solution:
   \[
   \begin{align*}
   m^2 + 3m + 2 &= 0 \\
   (m+1)(m+2) &= 0 \\
   y &= c_1 e^{-x} + c_2 e^{-2x}
   \end{align*}
   \]

2. In the "Big D" notation (*), becomes
   \[
   (D^2 + 3D + 2) [y] = 0 \quad L[y] = 0
   \]
   \[
   (D+1)(D+2) [y] = 0
   \]

a. Technically, \((D+1)(D+2)[y]\) means
   \[
   = (D+1) \left[ (D+2) [y] \right]
   = (D+1) \left[ D[y] + 2y \right] = (D+1) \left[ y' + 2y \right]
   = (D+1) [y'] + 2(D+1) [y]
   = (y'' + y') + 2(y' + y)
   = y'' + 3y' + 2y
   \]

b. The really neat thing is that if all the coefficients are constants, then the factors commute:
   \[
   (D+1)(D+2) [y] = 0
   \]
   \[
   (D+2)(D+1) [y] = 0
   \]
   see p 54 of these notes.
The Idea of an Annihilator Operator.

1. Find an operator \( L \) such that \( L[x^2] = 0 \)

   Try
   \[
   D^2[x^2] = DD[x^2] = D[2x] = 2
   \]
   \[
   D^2[x^2] = 2
   \]
   
   So \( D^3[x^2] = 0 \)

2. Find \( L \) s.t. (such that)

   \[
   L[x^2+2x-7] = 0
   \]

   Ans:
   \[
   \]

3. \( L = D^2 + 1 \)

   \[
   L[sin(x)] = (D^2+1)[sin(x)]
   \]
   \[
   = D^2[sin(x)] + sin(x)
   \]
   \[
   = D[cos(x)] + sin(x)
   \]
   \[
   = -sin(x) + sin(x) = 0
   \]

\[
\therefore D^2 + 1 \text{ annihilates } y = sin(x).
\]

AFTER CLASS

Homework: NB: § 4.5: p. 161: #1,5,11,15,19,23,29,39... (8)
Verification that \((D+1)(D+2) [y]\) "commutes."

(a) \((D+1)(D+2) [y]\)

\[
= (D+1) [(D+2) [y]] = (D+1) [y' + 2y] \\
= D [y' + 2y] + (y' + 2y) \\
= (y'' + 2y') + (y' + 2y) \\
= y'' + 3y' + 2y
\]

and

(b) \((D+2)(D+1) [y]\)

\[
= (D+2) [(D+1) [y]] = (D+2) [y' + y] \\
= D [y' + y] + 2(y' + y) \\
= (y'' + y') + 2y' + 2y \\
= y'' + 3y' + 2y
\]

And so you see that at least in this example the operators \((D+1)\) and \((D+2)\) commute.