I returned TEST #3a & NOTEBOOK #3a.

§7.4: p. 308: CONVOLUTIONS.

A. Definition: If \( f \) and \( g \) are piecewise continuous on \( [0, \infty) \), the convolution of \( f \) and \( g \), written \( f \ast g \), is defined by

\[
f \ast g (t) = \int_{\tau=0}^{\tau=t} f(\tau) g(t-\tau) \, d\tau
\]

B. "\( \ast \)" is commutative, i.e., \( f \ast g = g \ast f \)

Proof: \( \Box \)

\[
f \ast g (t) = \int_{\tau=0}^{\tau=t} f(\tau) g(t-\tau) \, d\tau \tag{A}
\]

2. And let \( u = t - \tau \), then \( du = -d\tau \)

3. And \( \tau = t - u \). If \( \tau = 0 \), \( u = t \); if \( \tau = t \), \( u = 0 \)

4. Now substitute into (A)

\[
\int_{\tau=0}^{\tau=t} f(\tau) g(t-\tau) \, d\tau = - \int_{u=0}^{u=t} f(t-u) g(u) \, du
\]

\[
= \int_{u=0}^{u=t} g(u) f(t-u) \, du = g \ast f
\]

5. \( \Box \). \( f \ast g = g \ast f \)

What Good is This "Convolution"?

Convolution Theorem (7.9) p 309.

If \( f \) and \( g \) are piecewise continuous on \( [0, \infty) \) and of exponential order, then

\[
\mathcal{L}\{ f \ast g \} = \mathcal{L}\{ f(t) \} \cdot \mathcal{L}\{ g(t) \} = F(s) \cdot G(s)
\]
Inverse Form of Convolution, p. 310.
\[ \mathcal{L}^{-1} \left\{ \frac{1}{(s-1)(s+4)} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s-1} \cdot \frac{1}{s+4} \right\} \]

Example of Usage (p. 310): Find, using, Inv. Conv.
\[ \mathcal{L}^{-1} \left\{ \frac{1}{(s-1)(s+4)} \right\} \]

Solution:
1. \[ \mathcal{L}^{-1} \left\{ \frac{1}{(s-1)(s+4)} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s-1} \cdot \frac{1}{s+4} \right\} \]
   = \[ e^t \ast e^{-4t} \]
   = \[ \int_{\tau=0}^{t} e^\tau \cdot e^{-4(t-\tau)} \, d\tau = \int_{\tau=0}^{t} e^{5\tau-4t} \, d\tau \]
   = \[ \left. \frac{e^{5\tau-4t}}{5} \right|_{\tau=0}^{t} = \frac{1}{5} \{ e^t - e^{-4t} \} \]

2. \[ \mathcal{L}^{-1} \left\{ \frac{1}{(s-1)(s+4)} \right\} = \frac{1}{5} \{ e^t - e^{-4t} \} \]

Check: Find \( \mathcal{L}^{-1} \) the "old way."

\[ \frac{1}{(s-1)(s+4)} = \frac{\sqrt{5}}{s-1} + \frac{-\sqrt{5}}{s+4} = \frac{1}{5} \left[ \frac{1}{s-1} - \frac{1}{s+4} \right] \]

\[ \mathcal{L}^{-1} \left\{ \frac{1}{(s-1)(s+4)} \right\} = \frac{1}{5} \{ e^t - e^{-4t} \} \]

See! \((\#)\) and \((\#\#)\) are THE SAME!
Theorem 7.10, p. 311.

If \( f(t) \) is piecewise continuous on \([0, \infty)\)
and of Exp.Order, and if \( f(t) \) is periodic
of period \( T \), then

\[
\mathcal{L}\{ f(t) \} = \frac{1}{1 - e^{-sT}} \int_{t=0}^{t=T} e^{-st} f(t) \, dt
\]

(Read the Proof in the Book)