SECTION 2.5:  LINEAR ORDINARY DIFFERENTIAL EQUATIONS (LODEs)

FIRST ORDER:

I. A First Order LODE looks like this (or it can be massaged into looking like this):

\[ a_1(x) \frac{dy}{dx} + a_0(x) y = g(x). \]  \hspace{1cm} (1.1)

Now if we are careful and restrict ourselves to intervals of \( x \) where \( a_1(x) \neq 0 \), we can divide both sides of (1.1) by \( a_1(x) \) getting:

\[ \frac{dy}{dx} + \frac{a_0(x)}{a_1(x)} y = \frac{g(x)}{a_1(x)}. \]  \hspace{1cm} (1.2)

And to avoid the slightly ugly-named coefficient function and "forcing function," we can re-name them as follows:

\[ \frac{dy}{dx} + P(x) y = f(x). \]  \hspace{1cm} (1.3)

where, of course, \( y \) is some as-yet-unknown function of \( x \) — this is what we are "solving for."

II. I like to call equations of the form (1.3) **Standard Form First Order LODEs.** And the solution to these equations is what we are going to master today.

It turns out that there are lots of 1-st Order LODEs out there to be solved, so this method is not unimportant.

In fact, I've already taught you the essentials of the "method" of solving 1-st Order LODEs. I just haven't given it a name. The method is called "using an integrating factor (IF)." And in these problems the integrating factor is the function \( e^{\int P(x) \, dx} \) where \( I(x) = \int P(x) \, dx \), and the whole thing is named \( \mu(x) \). Thus,

\[ \mu(x) = e^{\int P(x) \, dx} \hspace{1cm} \text{In } \int P(x) \, dx \text{ the } "\! C\! " \text{ is } \text{Zero} \]  \hspace{1cm} (1.4)

III. The key idea is to be able to compress the LHS of the LODE.

So here's how it works.

1. You get your equation into Standard Form (1.3).
2. You look for a short-cut. If there is, then go with it. If not, then proceed as follows:

3. You look at your $P(x)$.

4. You take the antiderivative of $P(x)$. I am calling it $I(x)$.

5. You raise $e$ to that power, getting $\mu(x)$. This is called "mu of x."

6. You multiply b.s. (both sides) of (1.3) by your $\mu(x)$.

7. You "compress" the LHS of the resulting eqn.

8. You integrate b.s.

9. You divide b.s. by $\mu(x)$.

10. You box your answer.

IV. Let's try some examples:

1. $y' + \frac{1}{x} y = 0$

2. $y' + 8y = 1$

3. $y' + 8y = 0$

4. $y' + 2xy = x^3$

5. $x^2 y' + 2xy = x^3$

6. $x^2 y' + 2xy = 0$
Examples.

1. \( y' + \frac{1}{x} y = 0 \)

Solution:

\[ P(x) = \frac{1}{x}; \quad I(x) = \int P(x) \, dx = \int \frac{1}{x} \, dx = \ln |x| \quad \text{Require} \ x > 0 \]

\[ = \ln x \quad \therefore \quad \mu(x) = e^{\int P(x) \, dx} = e^{\ln x} = x \]

\[ \mu(x) = x \]

\[ \boxed{b} \quad x(y' + \frac{1}{x} y) = x \cdot 0 = 0 \]

\[ \therefore \quad x y' + y = 0 \]

\[ \boxed{c} \quad \text{compress LHS} \quad (xy)' = 0 \]

Check by de-compression via "product rule"

\[ LHS = (xy)' = x \cdot y' + y \cdot 1 = xy' + y \]

\[ \frac{d}{dx} (x \cdot y(x)) = x \cdot \frac{dy}{dx} + 1 \cdot y(x) \]

\[ \boxed{d} \quad \text{Take anti. of b.s. of} \quad (xy)' = 0 \]

\[ xy = C \]

\[ \boxed{e} \quad y = \frac{C}{x} \]
\[ 2 \quad y' + 8y = 1 \]

**Solution**

\[ a \quad P(x) = 8; \quad I(x) = 8x; \quad \mu(x) = e^{8x} \]

\[ b \quad e^{8x}y' + 8e^{8x}y = e^{8x} \quad (\text{mult. b. o. by } \mu(x)) \]

\[ c \quad (e^{8x}y)' = e^{8x} \]

\[ d \quad e^{8x}y = \frac{e^{8x}}{8} + C \]

\[ e \quad y = \frac{1}{8} + Ce^{-8x} \]

\[ 3 \quad y' + 8y = 0 \]

**Solution**

\[ \ldots \quad e^{8x}y' + 8e^{8x}y = 0 \]

\[ \therefore (e^{8x}y)' = 0 \]

\[ \therefore e^{8x}y = C \]

\[ \therefore y = Ce^{-8x} \]

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end 10:45
\[4\] \quad y' + 2xy = x^3 \quad (\ast)

This one is kind of tricky!

\[\text{Soln:} \quad \boxed{a} \quad P(x) = 2x, \quad I(x) = \int P(x) \, dx = \int 2x \, dx = x^2 \quad (c = 0)
\]

\[\text{and} \quad \mu(x) = e^{x(x)} = e^{x^2} \quad \boxed{\mu(x) = e^{x^2}}\]

\[\boxed{b} \quad \text{Mult. b.s. of (\ast) w/} \quad \mu(x). \]

\[e^{x^2}y' + 2xe^{x^2}y = x^3e^{x^2}\]

\[\boxed{c} \quad \text{"Collapse" LHS:} \quad (e^{x^2}y)' = x^3e^{x^2} \quad (\ast\ast)\]

\[\text{CHECK the "collapse." by expanding}
\]
\[\frac{d}{dx} [e^{x^2}y(x)] = e^{x^2} \frac{d}{dx} y(x) + y(x) \cdot \frac{d}{dx} (e^{x^2})
\]
\[= e^{x^2}y' + 2xe^{x^2}y = e^{x^2} (y' + 2xy) \quad \checkmark\]

\[\boxed{d} \quad \text{Integrate b.s. of (\ast\ast)}
\]
\[e^{x^2}y = \int x^3 e^{x^2} \, dx\]

Now the problem becomes integrating the RHS.

... to be continued.