I Background for § 4.1.2 (p. 115)

Linear Dependence (LD) and Linear Independence (LI).

A We know \( \hat{i}, \hat{j}, \hat{k} \) vectors from Physics.
Any point in 3-space, e.g. \((1,2,3)\), can be expressed as a vector \( (1,2,3) \) or
\( 1 \hat{i} + 2 \hat{j} + 3 \hat{k} \).

B \( \hat{i}, \hat{j}, \hat{k} \) are "unit vectors" in that each is 1 unit in length \( \| \hat{i} \| = 1 \), etc.

C In College Alg you studied lines (straight lines).

1 \( y = mx + b \leftarrow y\)-int. FORM S-I,
\[ \hat{\text{slope}} \]

2 Gen (Std) Form \( ax + by = c \)

3 What about lines passing through the origin?

a S-I Form: \( y = mx \)

b Std Form: \( ax + by = 0 \)
\[ \text{Ex} \quad 2x + 3y = 0 \]

II Def. 4.1: p. 115 — LINEAR DEPENDENCE

A set of "n" functions \( f_1(x), f_2(x), \ldots, f_n(x) \) is said to be
LINEARLY DEPENDENT on an interval \( I \) iff there exist
constants, \( c_1, c_2, \ldots, c_n \) NOT ALL of which are ZERO, such that
\[ c_1 f_1(x) + c_2 f_2(x) + \ldots + c_{n-1} f_{n-1}(x) + c_n f_n(x) = 0 \]
Example: Let \( f_1(x) = x^2 \), \( f_2(x) = 2x^2 + 1 \), \( f_3(x) = 5 \).

**Question:** Do these 3 fcts. form a linearly dependent set?

**Solution:**

1. Set up a LC (linear combo.) & set it equal to zero.

\[
c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \quad (*)
\]

2. I know that \( c_1 = c_2 = c_3 = 0 \), then eq. (\( * \)) is TRUE!

3. The issue is, do there exist constants \( c_1, c_2, c_3 \) **not all zero**, for which (\( * \)) is TRUE?

4. If the answer to \( 3 \) is "YES" \( \Rightarrow \) I can display the \( c_1, c_2, c_3 \) —

Then I can say \( f_1(x), f_2(x), \) and \( f_3(x) \)

are **LINEARLY DEPENDENT**.

\[
c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0
\]

\[
c_1(x^2) + c_2(2x^2 + 1) + c_3(5) = 0
\]

\[
(c_1 + 2c_2)x^2 + (c_2 + 5c_3) = 0 \quad (**)
\]

**Remember This**

If \( ax^2 + bx + c = dx^2 + ex + g \), then

I know that \( a = d \Rightarrow b = e \Rightarrow c = g \)

"Two Polynomials are equal iff their corresponding coefficients are equal."

So the coef. of \( x^2 \) on the left side of (\( ** \)) is \( c_1 + 2c_2 \)

and the coef. of \( x^2 \) on the right side of (\( ** \)) is \( 0 \)

\[ \begin{align*}
\text{etc.} & \quad \Rightarrow \text{From (\( ** \)) I conclude} \\
& \quad c_1 + 2c_2 = 0 \\
& \quad c_2 + 5c_3 = 0
\end{align*} \]

(cont....)
\[ \begin{align*}
\begin{cases}
c_1 + 2c_2 = 0 \\
c_2 + 5c_3 = 0
\end{cases} \Rightarrow c_2 = -\frac{1}{2} c_1 \\
\therefore c_1 = -2c_2 \quad \text{and} \quad c_3 = -\frac{1}{5} c_2
\end{align*} \]

Pick \( c_2 = -5 \) \( \therefore c_1 = 10 \) \( \therefore c_3 = 1 \).

My claim is that

\[ 10(x^2) - 5(2x^2 + 1) + 1(5) = 0 \]

\( \therefore x^2, 2x^2 + 1, 5 \) are L.D. on the interval \( I = (-\infty, \infty) \).

Another way to look at linear dependence is this: in the example above, for instance, \( \textbf{ANY ONE} \) of the 3 functions can be written "in terms of" the \( \textbf{OTHER TWO} \)!

For example

\[ f_1(x) = 5f_2(x) - f_3(x) \]

\( (\text{Because} \quad x^2 = 5(2x + 1) - 1(5)) \)