No Tests to Return Today.

BEGIN UNIT #3

Study §4.5: Differential Operators (p. 157)

A Background.

1. Functions. A function is a rule of association that relates an input variable with an output variable.

$$f: x \mapsto y$$

The metaphor

- **LHS**: Label
- **RHS**: Rule

Label := something

In algebra (not college alg. MAC 1105)

$$f: x \mapsto \text{rule}$$

Ex. $$f: x \mapsto \cosh x$$ Sometimes

$$x \mapsto \cosh x$$ (one output value at most).

2. What about a function that "acts" upon functions, rather than numbers?

The operation of "taking a derivative" is a function acting upon functions.

New Notation: D

Use of new notation: $$D[x^2] = 2x$$

$$D: x^2 \mapsto 2x$$

D is a "differential operator."
Differential Operators.

1. \( D = \frac{d}{dx} = f' \) \( [ f' \] means "is identical with" or "is defined" \( f \)

\[ D[\sin x + e^{-2x}] = \cos x - 2e^{-2x} \]

For example.

2. Well what about \( D^2 \) ? \( D^2 = D(D) \)

\[ D^2[\sin x] = D(D[\sin x]) = D[\cos x] \]

\[ D^2[\sin x] = -\sin x \]

(ii) Note: The "derivative idea" is a "linear idea":

\[ \frac{d}{dx}(c_1f + c_2g) = c_1 \frac{df}{dx} + c_2 \frac{dg}{dx} \]

3. \( D[c_1f + c_2g] = c_1 D[f] + c_2 D[g] \).

Also \( (D^2 + D)[e^{-2x}] = D^2[e^{-2x}] + D[e^{-2x}] \)

\[ = 4e^{-2x} - 2e^{-2x} = 2e^{-2x} \]

4. Note: \( D^0[f] = I[f] = f \)

often \( D^0[f] = \square f \)

Example \( (D^2 + 3D + 2)[y] \)

\[ = y'' + 3y' + 2y \]

5. So the ODE \( y'' + 3y' + 2y = 0 \) \( (*) \)

Can be written as \( (D^2 + 3D + 2)[y] = 0 \) \( (**) \)

To solve \( (*) \)

\[ \begin{align*}
1. & m^2 + 3m + 2 = 0 \\
2. & (m + 1)(m + 2) = 0 \\
3. & y = c_1e^{-x} + c_2e^{-2x}
\end{align*} \]

(Cont...)
To Solve (***)

1. \(D^2 + 3D + 2\)

\((D + 2)(D + 1)\) and "go from there"

Note \((D+2)(D+1) = (D+1)(D+2)\)

This DiffOp. is commutative

Look at this \((D+2)(D+1)\ [e^{-x}]\)

**Solution**

\((D+2)\ [(D+1)\ [e^{-x}]]\)

\((D+2)\ [-e^{-x} + e^{-x}] = (D+2)\ [0] = 0 \checkmark\)

Compare w/ 

\((D+1)\ [(D+2)\ [e^{-x}]] = (D+1)\ [-e^{-x} + 2e^{-x}]\)

\[= (D+1)\ [e^{-x}] = -e^{-x} + 1 \cdot e^{-x} = 0 \checkmark\]

6. For shorthand, we can use the "L" notation.

For example, if I say (define)

\[L = D^2 + 3D + 2\]

then

\[L[y] = (D^2 + 3D + 2)[y] \text{ and} \]

\[L[y] = 0 \text{ means } (D^2 + 3D + 2)[y] = 0 \]

\[\text{means } y'' + 3y' + 2y = 0\]

7. I want to look for **ANNIHILATORS**!

What is an annihilator?

Well — you start w/ a function and you want to annihilate it.

Given a function I'd like to find some Linear DiffOp., \(L\), which has the property that

\[L[y] = 0\]

(cont...)
Example: $y = e^{2x}$ and I wish to annihilate it.

I have to "dream-up" a DiffOp, $L$, such that

$$L[e^{2x}] = 0$$

[This is not zero, the number; this is zero, the function, i.e. this is the x-axis, graphically.]

I'm just going to show you the "answer" now, and we'll learn how to actually "get" the answer presently.

Let $L[y] = (D-2)[y]$.

Verify that $L$ annihilates $y = e^{2x}$.

Verification: $L[e^{2x}] = (D-2)[e^{2x}] = 2e^{2x} - 2e^{2x} = 0$

$\therefore L[e^{2x}] = 0$ and, by definition, $L$ annihilates $e^{2x}$. 

END