3.1 The derivative as a function

**Def:** The derivative of the function \( f(x) \) with respect to the variable \( x \) is the function \( f' \) whose value \( x \) is

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \text{ provided the limit exists.}
\]

\[
f'(x) = \frac{df}{dx} = \frac{d}{dx} f(x) = D(f)(x) = D_x f(x)
\]

This means that the derivative at a value \( x_0 \) is the slope of the tangent line at \( x_0 \). See 2.7

1. \( k(z) = \frac{1-z}{z} \); Find \( k'(z) \).

\[
k'(z) = \lim_{h \to 0} \frac{k(z+h) - k(z)}{h} = \lim_{h \to 0} \frac{-h}{2z(z+h)} \left( \frac{1}{h} \right)
\]

\[
= \lim_{h \to 0} \frac{-1}{2z(z+h)}
\]

\[
= \frac{-1}{2z^2} = k'(z)
\]

\[
k'(-1) = \frac{-1}{2}
\]

\[
k'(1) = \frac{-1}{2}
\]

\[
k'(\sqrt{2}) = \frac{-1}{4}
\]

Aside: \( k(z+h) - k(z) \)

\[
= \frac{1-(z+h)}{2(z+h)} - \frac{1-z}{2z}
\]

\[
= \frac{z(1-z-h)-(z+h)(1-z)}{2z(z+h)}
\]

\[
= \frac{z-z^2-hz-(z-z^2-hz)}{2z(z+h)}
\]

\[
= \frac{z-\sqrt{2} - h(z-\sqrt{2} - h)}{2z(z+h)}
\]

\[
\frac{-h}{2z(z+h)} \text{ plug back in}
\]
Find \( \frac{dz}{dw} \) if \( z = \frac{1}{\sqrt{3w-2}} = z(w) \)

\[
\frac{dz}{dw} = \lim_{h \to 0} \frac{z(h+w) - z(w)}{h}
\]

Aside: \( z(h+w) - z(w) \)

\[
= \frac{1}{\sqrt{3(h+w)-2}} - \frac{1}{\sqrt{3w-2}}
\]

\[
= \frac{\sqrt{3w-2} - \sqrt{3h+3w-2}}{\sqrt{3h+3w-2} \sqrt{3w-2}}
\]

\[
= \frac{3w-2 - (3h+3w-2)}{(\sqrt{3h+3w-2}(\sqrt{3w-2} + \sqrt{3h+3w-2}))}
\]

\[
= \frac{3w-2 - 3h - 3w + 2}{(\sqrt{3h+3w-2}(\sqrt{3w-2} + \sqrt{3h+3w-2}))}
\]

\[
\frac{dz}{dw} = \lim_{h \to 0} \frac{-3h}{(\sqrt{3h+3w-2}(\sqrt{3w-2} + \sqrt{3h+3w-2}))} \left( \frac{1}{h} \right)
\]

\[
= \frac{-3}{(\sqrt{3w-2}(\sqrt{3w-2} + \sqrt{3w-2}))} = \frac{-3}{(3w-2)(2\sqrt{3w-2})} = \]
\[
\frac{-3}{2(3w-2)^{3/2}} = \frac{-3}{2(3w-2)^{3/2}}
\]

II. Differentiable on an Interval; one-sided derivatives.

A function \( y = f(x) \) is **differentiable** on an open interval (finite or infinite) if it has a derivative at each point on the interval. It is **differentiable** on a closed interval \([a, b]\) if it is differentiable on the interior \((a, b)\) and if the limits

\[
\lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h} \quad \text{Right-hand derivative at } a
\]

\[
\lim_{h \to 0^-} \frac{f(b + h) - f(b)}{h} \quad \text{Left-hand derivative at } b
\]

exist at the endpoints.

Note: A function does **not** have a derivative at a point?

1. A **corner**
2. A **cusp**
3. A **vertical tangent**
4. A **discontinuity** (jump, hole, asymptote, not in domain)
A Theorem: Differentiability Implies Continuity

If \( f \) has a derivative at \( x = c \), then \( f \) is continuous at \( x = c \).

B The Intermediate Value Property of Derivatives

**Theorem: Darboux's Theorem.**

If \( a \) and \( b \) are any two points in an interval on which \( f \) is differentiable, then \( f' \) takes on every value between \( f'(a) \) and \( f'(b) \).

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1. **Show that function is not differentiable at \( c \).**

\[
\lim_{h \to 0^-} \frac{f(1+h)-f(1)}{h} = \text{left function is } y = x
\]

\[
\lim_{h \to 0^-} \frac{1+h-1}{h} = 1
\]

\[
\lim_{h \to 0^+} \frac{f(1+h)-f(1)}{h} = \text{right function is } y = \frac{1}{x}
\]

\[
\lim_{h \to 0^+} \frac{1}{1+h} - \frac{1}{h} = \lim_{h \to 0^+} \frac{1}{h} - \frac{1}{h} = -1
\]

\* Since from left does not equal from right, derivative does not exist.
At what domain points does the function appear to be:

a) Differentiable?

\([-2, -1) \cup (-1, 0) \cup (0, 2) \cup (2, 3]\)

b) Continuous but not differentiable?

\(x = -1\)

c) Neither continuous nor differentiable.

\(x = 0, 2\)