3.1 Related Rates

- Positive rate - increasing
- Negative rate - decreasing

1. Draw picture
2. Label variables. Let $t = \text{time}$
3. Write down equation
4. Differentiate with respect to $t$
5. Solve for unknown.

1. Suppose that the radius $r$ and surface area $S = 4\pi r^2$ of a sphere are differentiable functions of $t$. Write an equation that relates $dS/dt$ to $dr/dt$.

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

2. The radius $r$ and height $h$ of a right circular cone are related to the cone's volume $V$ by the equation $V = \frac{1}{3}\pi r^2 h$.

a) How is $dV/dt$ related to $dh/dt$ if $r$ is constant? (i.e. pretend $r$ is a number when taking derivative)

$$\frac{dV}{dt} = \frac{1}{3} \pi r^2 \frac{dh}{dt}$$

(Did not take derivative of $r$)

b) How is $dh/dt$ related to $dV/dt$ if $h$ is constant? (Pretend $h$ is just a number)

$$\frac{dV}{dt} = \frac{2}{3} \pi r h \frac{dh}{dt}$$

(Did not take derivative of $h$)
c) How is $\frac{dV}{dt}$ related to $\frac{dx}{dt}$ and $\frac{dh}{dt}$ if neither \( r \) nor \( h \) is constant?

\[
V = \frac{1}{3} \pi r^2 h
\]

\[
\frac{dV}{dt} = \frac{1}{3} \pi r^2 \frac{dh}{dt} + h \left( \frac{2}{3} \pi r \frac{dr}{dt} \right)
\]

\[
\frac{dV}{dt} = \frac{1}{3} \pi r^2 \frac{dh}{dt} + \frac{2}{3} \pi r h \frac{dr}{dt}
\]

3) If \( x, y, \) and \( z \) are lengths of the edges of a rectangular box, the common length of the box's diagonals is

\[
d = \sqrt{x^2 + y^2 + z^2}
\]

c) Assuming that \( x, y, \) and \( z \) are differentiable functions of \( t \), how is \( \frac{ds}{dt} \) related to \( \frac{dx}{dt}, \frac{dy}{dt}, \) and \( \frac{dz}{dt} \)?

\[
\frac{ds}{dt} = \frac{1}{2} (x^2 + y^2 + z^2)^{-\frac{1}{2}} \left( 2x \frac{dx}{dt} + 2y \frac{dy}{dt} + 2z \frac{dz}{dt} \right)
\]

c) How is \( \frac{ds}{dt} \) related to \( \frac{dx}{dt} \) and \( \frac{dz}{dt} \) if \( x \) is constant?

(differentiate \( x \) as just a number)

\[
\frac{ds}{dt} = \frac{1}{2} (x^2 + y^2 + z^2)^{-\frac{1}{2}} \left( 0 + 2y \frac{dy}{dt} + 2z \frac{dz}{dt} \right)
\]

\[
\frac{ds}{dt} = \frac{1}{2} (x^2 + y^2 + z^2)^{-\frac{1}{2}} \left( 2y \frac{dy}{dt} + 2z \frac{dz}{dt} \right)
\]

c) How are \( \frac{dx}{dt}, \frac{dy}{dt}, \) and \( \frac{dz}{dt} \) if \( s \) is constant?

(differentiate \( s \) as just a number)

\[
0 = \frac{1}{2} (x^2 + y^2 + z^2)^{-\frac{1}{2}} \left( 2x \frac{dx}{dt} + 2y \frac{dy}{dt} + 2z \frac{dz}{dt} \right)
\]
4. When a circular plate of metal is heated in an oven, its radius increases at a rate of .01 cm/min. At what rate is the plate's area increasing when the radius is 50 cm?

\[
\frac{dr}{dt} = .01 \text{ cm/min} \quad r = 50 \text{ cm} \quad \frac{dA}{dt} = ?
\]

\[A = \pi r^2\]

\[
\frac{dA}{dt} = 2\pi r \frac{dr}{dt}
\]

\[
\frac{dA}{dt} = 2\pi (50)(.01) = \pi = 3.14 \text{ cm}^2/\text{min}.
\]

5. The mechanics at Lincoln Automotive are reboring a 6 in deep cylinder to fit a new piston. The machine they are using increases the cylinder's radius one-thousandth of an inch every 3 min. How rapidly is the cylinder volume increasing when the bore (diameter) is 3.8 in?

\[V = \pi r^2 h\]

\[
r \text{ changes but } h \text{ is constant}
\]

\[
\frac{dV}{dt} = 12\pi r \frac{dr}{dt}
\]

\[
\frac{dV}{dt} = 12\pi (1.9)(.001) = .024 \text{ in}^3/\text{min}
\]
You are videotaping a race from a stand 132 ft from the track, following a car that is moving at 180 m/hr (264 ft/sec). How fast will your camera angle $\theta$ be changing when the car is right in front of you? A half second later?

Picture given

$\tan \theta = \frac{x}{132}$

$\frac{dx}{dt} = 264 \text{ ft/sec}$

$132 \tan \theta = x$

$132 (\sec^2 \theta) \frac{dt}{dt} = \frac{dx}{dt}$

$132 (\sec^2 \theta) \frac{d\theta}{dt} = 264 \text{ ft/sec}$

$\frac{d\theta}{dt} = \frac{264}{132 (\sec^2 \theta)} = \frac{2}{\sec^2 \theta}$

If car is right in front, then $\theta = 0$

Hence $\frac{d\theta}{dt} = \frac{2}{\sec^2 \theta} = \frac{2 \text{ rad}}{\text{sec}}$

A half second later, $x = 264 \times (1/2 \text{ sec})$

$x = 132 \text{ ft}$

$\tan \theta = \frac{132}{132} = 1$

$\frac{d\theta}{dt} = \frac{2}{\sec^2 (\theta)} = \frac{2}{2} = 1 \text{ rad/sec}$

$\theta = \frac{\pi}{4}$
Coffee is draining from a conical filter into a cylindrical coffee pot at the rate of 10 in³/min.

a) How fast is the level in the pot rising when the coffee in the cone is 5 in deep?

b) How fast is the level in the cone falling then?

\[ \frac{dh_c}{dt} = \frac{10}{9\pi} \approx 0.354 \text{ in/min} \]

**Picture given**

**Facts**

\[ \frac{dV_c}{dt} = -10 \text{ in}^3/\text{min} \quad \frac{dV_f}{dt} = 10 \text{ in}^3/\text{min} \]

- \( h_c = 5 \) in
- \( r_c = \frac{h_c}{2} \)

- \( p \) - pot
- \( c \) - conical
- \( V \) - volume
- \( h \) - height of coffee
- \( r \) - radius

**Conical Filter**

\[ V_c = \frac{1}{3} \pi r_c^2 h_c \]

Volume, radius, and height change.

we know \( \frac{dh}{dt} \) and want \( \frac{dr}{dt} \).

Since we do not know \( \frac{dr}{dt} \), we need to change \( r \) in terms of \( h \) to consider the change.

\[ V_c = \frac{1}{12} \pi h_c^3 \]

\[ \frac{dV_c}{dt} = \frac{1}{4} \pi h_c^2 \frac{dh_c}{dt} \]

\[ -10 \text{ in}^3/\text{min} = \frac{1}{4} \pi (5)^2 \frac{dh_c}{dt} \]

\[ -10 \left( \frac{4}{2\pi} \right) = \frac{dV_c}{dt} \approx -3.59 \text{ in/min} \]