4.1 The Extreme Value of Functions

Def: Let $f$ be a function with domain $D$. Then $f$ has an absolute maximum value on $D$ at a point $c$ if $f(x) \leq f(c)$ for all $x$ in $D$. (biggest $y$-value)
and an absolute minimum value on $D$ at $c$ if $f(x) \geq f(c)$ for all $x$ in $D$. (smallest $y$-value)

* The Extreme Value Theorem

If $f$ is continuous on a closed interval $[a,b]$, then $f$ attains both an absolute maximum value $M$ and an absolute minimum value $m$ in $[a,b]$. That is, there are numbers $x_1$ and $x_2$ in $[a,b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other $x$ in $[a,b]$

($M$ - largest $y$-value for any $x$ in $[a,b]$)
($m$ - smallest $y$-value for any $x$ in $[a,b]$)

$\exists x_1 \in [a,b] \text{ such that } f(x_1) = M$
$\exists x_2 \in [a,b] \text{ such that } f(x_2) = m$

$M$ (absolute max) can appear more than once.
$m$ (absolute min) can appear " " ".

* Does not always work with open or half-open intervals like $(a,b)$, $[a,b)$ or $(a,b]$ or discontinuity existing.
Def. The $x$-values where $f(x) = 0$ and/or undefined are called critical values.

To Find Absolute Extrema of a continuous function $f$ on a finite closed interval:

1. Evaluate $f$ at all critical points and endpoints.
2. Take largest and smallest value.

**Ex.** $f(x) = -x - 4, \ -4 \leq x \leq 1$ Find $M$ and $m$. Graph.

- $f'(x) = -1 \neq 0$
- $f(-4) = 0 \leq M$
- $f(1) = -5 \leq m$
- Largest $y, M = 0$
- Smallest $y, m = -5$

**Ex.** $g(x) = -\sqrt{5} - x^2, \ -\sqrt{5} \leq x \leq 0$ Find $M$ and $m$. Graph.

- $g'(x) = \frac{1}{2} (5-x^2)^{-\frac{1}{2}} (-2x)$
- $g'(x) = \frac{x}{\sqrt{5-x^2}}$

Critical points

$$\frac{x}{\sqrt{5-x^2}} = 0$$

$x = 0, \ g(0) = -\sqrt{5}$

undefined at $x = \pm \sqrt{5}$

**End points**

$g(-\sqrt{5}) = 0 \leq \text{absolute maximum}$

$g(0) = -\sqrt{5} \leq \text{absolute minimum}$

$g(\sqrt{5}) = 0$

Graph $g(x) = -\sqrt{5} - x^2$

Use calculator as aid
**Ex**

M does not exist (largest y-value)

\[ m = 0 \] (smallest y-value)

**Ex**

\[ M = 5 \]  
\[ m \text{ does not exist.} \]

**Defn.** A function \( f \) has a **local minimum** value at an interior point \( c \) of its domain if \( f(x) \geq f(c) \) for all \( x \) in some open interval containing \( c \).

- Local min
- Rel. min
- Anywhere graph takes this shape, you have rel. min.

A function \( f \) has a **local maximum** value at an interior point \( c \) of its domain if \( f(x) \leq f(c) \) for all \( x \) in some open interval containing \( c \).

- Local max
- Rel. max
- Anywhere graph takes this shape, you have rel. max.

**Theorem:** The **First Derivative Theorem** for Local Extreme Values. If \( f \) has a local max or min value at \( c \) in domain and \( f'(c) \) exists, then

\[ f'(c) = 0 \]

(That is, the slope of tangent lines at max or mins is 0)

*Always remember derivative is slope*
Finding local extrema

1. Find critical points, set $y' = 0$ and where $y'$ is undefined.
2. Determine the local extrema.

Example: $y = x^{\frac{2}{3}}(x^2 - 4)$

$y' = x^{\frac{2}{3}}(2x) + (x^2 - 4)(\frac{2}{3}x^{-\frac{1}{3}})$

$y' = 2x^{\frac{5}{3}} + \frac{2}{3}x^2 - \frac{8}{3}x^{-\frac{1}{3}}$

$y' = \frac{6}{3}x^{\frac{5}{3}} + \frac{2}{3}x^{\frac{4}{3}} - \frac{8}{3}x^{-\frac{1}{3}}$

$y' = \frac{-8}{3}x^{\frac{5}{3}} - \frac{8}{3}x^{-\frac{1}{3}}$

$\frac{8}{3}x^{\frac{5}{3}} - \frac{8}{3}x^{-\frac{1}{3}} = 0$

$\frac{8}{3}x^{-\frac{1}{3}}(x^{\frac{5}{3}} - 1) = 0$

$\frac{8}{3}x^{-\frac{1}{3}} \neq 0 \quad x^{\frac{5}{3}} - 1 = 0$

$x = \pm 1$

Critical values at $x = \pm 1$ and undefined at $x = 0$

To find $y$-value plug into $y = x^{\frac{2}{3}}(x^2 - 4)$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = x^{\frac{2}{3}}(x^2 - 4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$-4$ Relative minimum values</td>
</tr>
<tr>
<td>$-1$</td>
<td>$-4$ Relative maximum values</td>
</tr>
<tr>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

$(1, 4)$ (at $x = 1$)

$(1, 4)$ (at $x = 1$)
Suppose that at any given time \( t \) (in seconds) the current \( i \) (in amperes) in an alternative current circuit is \( i = 2\cos t + 2\sin t \). What is the peak current for this circuit (largest magnitude)?

\[
\frac{d}{dt} i = -2\sin t + 2\cos t
\]

- \( -2\sin t + 2\cos t = 0 \) to find critical values.

\[
2\cos t = 2\sin t
\]

\[
\cos t = \sin t
\]

Reference angle is \( \frac{\pi}{4} \), \( \sin t = \cos t \) in Q I, III

Graph

Peaks

\[
i = 2\cos t + 2\sin t
\]
\[
i = 2\cos \frac{\pi}{4} + 2\sin \frac{\pi}{4}
\]
\[
i = \left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}
\]

\[\text{largest}\]

Peak current is \( 2\sqrt{2} \) amperes

\[\text{Smallest}\]

Derivative give potential candidates. Must plug back into original to find largest or smallest.